Cyclic Codes

Definition: A linear block code is cyclic if a cyclic shift of any codeword is another codeword.

The i-th shift of $\underline{v} = (v_0, v_1, \ldots, v_{n-1})$ is:

$$\underline{v}^{(i)} = (v_{n-i}, v_{n-i+1}, \ldots, v_{n-1}, v_0, v_1, \ldots, v_{n-i-1})$$

For example:

$$\underline{v}^{(1)} = (v_{n-1}, v_0, v_1, \ldots, v_{n-2})$$

and

$$\underline{v}^{(2)} = (v_{n-2}, v_{n-1}, v_0, v_1, \ldots v_{n-3}).$$

Example:

A (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

Messages	Code vectors	Code polynomials
(0000)	0000000	$0 = 0 \cdot \mathbf{g}(X)$
(1000)	1101000	$1 + X + X^3 = 1 \cdot \mathbf{g}(X)$
(0100)	0110100	$X + X^2 + X^4 = X \cdot \mathbf{g}(X)$
(1100)	1011100	$1 + X^2 + X^3 + X^4 = (1 + X) \cdot \mathbf{g}(X)$
(0010)	0011010	$X^2 + X^3 + X^5 = X^2 \cdot \mathbf{g}(X)$
(1010)	1110010	$1 + X + X^2 + X^5 = (1 + X^2) \cdot \mathbf{g}(X)$
(0110)	0101110	$X + X^3 + X^4 + X^5 = (X + X^2) \cdot \mathbf{g}(X)$
(1110)	1000110	$1 + X^4 + X^5 = (1 + X + X^2) \cdot \mathbf{g}(X)$
(0001)	0001101	$X^3 + X^4 + X^6 = X^3 \cdot \mathbf{g}(X)$
(1001)	1100101	$1 + X + X^4 + X^6 = (1 + X^3) \cdot \mathbf{g}(X)$
(0101)	0111001	$X + X^2 + X^3 + X^6 = (X + X^3) \cdot \mathbf{g}(X)$
(1101)	1010001	$1 + X^2 + X^6 = (1 + X + X^3) \cdot \mathbf{g}(X)$
(0011)	0010111	$X^2 + X^4 + X^5 + X^6 = (X^2 + X^3) \cdot \mathbf{g}(X)$
(1011)	1111111	$1 + X + X^2 + X^3 + X^4 + X^5 + X^6 = (1 + X^2 + X^3) \cdot \mathbf{g}(X)$
(0111)	0100011	$X + X^5 + X^6 = (X + X^2 + X^3) \cdot \mathbf{g}(X)$
(1111)	1001011	$1 + X^3 + X^5 + X^6 = (1 + X + X^2 + X^3) \cdot \mathbf{g}(X)$

Let

$$v(X) = v_0 + v_1 X + v_2 X^2 + \cdots + v_{n-1} X^{n-1}$$

be the polynomial representation of $\underline{v}$.

Then:

$$v^{(i)}(X) = v_{n-i} + v_{n-i+1} X + \cdots + v_{n-1} X^{i-1} + \cancel{v_0 X^i} + \cancel{v_1 X^{i+1}} + \cdots + \cancel{v_{n-i-1} X^{n-1}}$$

Multiply X^i by $v(X)$, i.e., shift $\underline{v}$ i times (linearly, not cyclically). Then

$$X^i v(X) = \cancel{v_0 X^i} + \cancel{v_1 X^{i+1}} + \cdots + \cancel{v_{n-i+1} X^{n-1}} + \cdots + v_{n-1} X^{n+i-1}$$

Add $X^i v(X)$ and $v^{(i)}(X)$:

$$X^i v(X) + v^{(i)}(X) = v_{n-i} + v_{n-i+1} X + \cdots + v_{n-1} X^{i-1} + v_{n-i} X^n + v_{n-i+1} X^{n+1} + \cdots v_{n-1} X^{n+i-1}$$

or

$$X^i v(X) + v^{(i)}(X) = [v_{n-i} + v_{n-i+1} X + \cdots v_{n-1} X^{i-1}](X^n+1)$$

So:

$$X^i v(X) = q(X)[X^n+1] + v^{(i)}(X)$$

That is the i-th cyclic shift of $\underline{v}(X)$ is generated by dividing $X^i v(X)$ by X^n+1.

Theorem 1: The non-zero code polynomial with minimum degree in a cyclic code ~~is~~ C is unique.

Proof: Let $g(x) = g_0 + g_1 x + \cdots g_{r-1} x^{r-1} + x^r$ be the minimal degree r code polynomial of C.

suppose there is another $g'(x) = g'_0 + g'_1 x + \cdots g'_{r-1} x^{r-1} + \cdots$

Then $g(x) + g'(x)$ ~~is~~ is another codeword in C with degree less than r. $\Rightarrow$ Contradiction.

Theorem 2: Let $g(x) = g_0 + g_1 x + \cdots g_{r-1} x^{r-1} + x^r$ be the minimum degree polynomial of a cyclic code C. Then $g_0 \neq 0$.

Proof: If $g_0 = 0$ then shifting $g(x)$ once to the left (or $r-1$ times to right) results in

$$g_1 + g_2 x + \cdots + g_{r-1} x^{r-2} + x^{r-1}$$

which has a degree $< r \Rightarrow$ Contradiction.

So $\quad g(x) = 1 + g_1 x + g_2 x^2 + \cdots + g_{r-1} x^{r-1} + x^r$

Let $g(X)$ be the polynomial of minimum degree of a code C.

Take $g(X), X g(X), X^2 g(X), \ldots, X^{n-r-1} g(X)$

These are shifts of $g(X)$ by $0, 1, \ldots, n-r-1$

So they are code words. Any linear combination of them is also a codeword, so:

$$v(X) = u_0 g(X) + u_1 X g(X) + \cdots + u_{n-r+1} X^{n-r-1} g(X)$$

$$= [u_0 + u_1 X + \cdots + u_{n-r-1} X^{n-r-1}] g(X)$$

is also a code.

Theorem 3: Let $g(X) = 1 + g_1 X + \cdots + g_{r-1} X^{r-1} + X^r$ be the non-zero code polynomial of minimum degree of an (n, k) cyclic code C. A binary polynomial of degree $n-1$ or less is a code polynomial if and only if it is a multiple of $g(X)$.

Proof: Let $v(X)$ be a polynomial of degree $n-1$ or less such that:

$$v(X) = (a_0 + a_1 X + \cdots a_{n-r-1} X^{n-r-1}) g(X)$$

Then:

$$v(X) = a_0 g(X) + a_1 X g(X) + \cdots a_{n-r-1} X^{n-r-1} g(X)$$

Since $g(X)$, $X g(X)$, ... are each codeword of C so is their sum $v(X)$. ✓

Now assume $v(X)$ be a code polynomial in C. Then write:

$$v(X) = a(X) g(X) + b(X)$$

i.e., divide $v(X)$ by $g(X)$ and get remainder $b(X)$ and quotient $a(X)$.

$$b(X) = v(X) + a(X) g(X)$$

$v(X)$ is a codeword and so is $a(X) g(X)$

Therefore, $b(X)$ is also a codeword. But degree of $b(X)$ is less than $r \Rightarrow$ contradiction unless if $b(X) = 0$.

The number of polynomials of degree $n-1$ or less that are multiple of $g(X)$ is 2^{n-r}.

Due to 1-to-1 correspondence between these polynomials and the codewords (Theorem 3), we have $2^{n-r} = 2^k \Rightarrow \boxed{r = n-k}$.

Theorem 4: In an (n,k) cyclic code, there is one and only one code polynomial of degree $n-k$,

$$g(X) = 1 + g_1 X + g_2 X^2 + \cdots + g_{n-k-1} X^{n-k-1} + X^{n-k}$$

Every code polynomial is a multiple of $g(X)$. Every binary polynomial of degree $n-1$ or less that is a multiple of $g(X)$ is a code polynomial.

So:

$$V(X) = U(X) g(X)$$

is a code polynomial, however, not in a systematic form.

To make code systematic, multiply the information polynomial $u(X)$ by X^{n-k}. This means placing the k information bits at the head of the shift register (in k right-most Flip-Flops). Then

$$u(X) = u_0 + u_1 X + \cdots + u_{k-1} X^{k-1}$$

will result in:

$$X^{n-k} u(X) = u_0 X^{n-k} + u_1 X^{n-k+1} + \cdots + u_{k-1} X^{n-1}$$

Now divide $X^{n-k}u(X)$ by $g(X)$ to get:

$$X^{n-k}u(X) = a(X)g(X) + b(X)$$

where $b(X)$ is a polynomial of degree $n-k-1$ or less:

$$b(X) = b_0 + b_1X + \cdots + b_{n-k-1}X^{n-k-1}$$

$$b(X) + X^{n-k}u(X) = a(X)g(X)$$

This means that $b(X) + X^{n-k}u(X)$ is the represadation of a codeword in systematic form; i.e.,

$$b(X) + X^{n-k}u(X) = b_0 + b_1X + \cdots + b_{n-k-1}X^{n-k-1} + u_0X^{n-k} + u_1X^{n-k+1} + \cdots + u_{k-1}X^{n-1}.$$

represents

$$\underline{v} = (b_0, b_1, \ldots, b_{n-k-1}, u_0, u_1, \ldots, u_{k-1})$$

<u>Example</u>: Consider the $(7,4)$ cyclic code generated by $g(X) = 1 + X + X^3$. Let $u(X) = 1 + X^3$.

Then:

1) $X^3u(X) = X^3 + X^6$

2)
$$\begin{array}{r|l}
 & X^3 + X \\ \hline
X^3 + X + 1 & X^6 + X^3 \\
 & X^6 + X^4 + X^3 \\ \hline
 & X^4 \\
 & X^4 + X^2 + X \\ \hline
 & X^2 + X \leftarrow b(X)
\end{array}$$

3) $V(X) = b(X) + X^3 u(X)$

$$= X + X^2 + X^3 + X^6$$

or

$$\underline{V} = (0, 1, 1, 1, 0, 0, 1)$$

A (7, 4) cyclic code in systematic form generated by $\mathbf{g}(X) = 1 + X + X^3$.

Message	Codeword	
(0000)	(0000000)	$0 = 0 \cdot \mathbf{g}(X)$
(1000)	(1101000)	$1 + X + X^3 = \mathbf{g}(X)$
(0100)	(0110100)	$X + X^2 + X^4 = X\mathbf{g}(X)$
(1100)	(1011100)	$1 + X^2 + X^3 + X^4 = (1 + X)\mathbf{g}(X)$
(0010)	(1110010)	$1 + X + X^2 + X^5 = (1 + X^2)\mathbf{g}(X)$
(1010)	(0011010)	$X^2 + X^3 + X^5 = X^2\mathbf{g}(X)$
(0110)	(1000110)	$1 + X^4 + X^5 = (1 + X + X^2)\mathbf{g}(X)$
(1110)	(0101110)	$X + X^3 + X^4 + X^5 = (X + X^2)\mathbf{g}(X)$
(0001)	(1010001)	$1 + X^2 + X^6 = (1 + X + X^3)\mathbf{g}(X)$
(1001)	(0111001)	$X + X^2 + X^3 + X^6 = (X + X^3)\mathbf{g}(X)$
(0101)	(1100101)	$1 + X + X^4 + X^6 = (1 + X^3)\mathbf{g}(X)$
(1101)	(0001101)	$X^3 + X^4 + X^6 = X^3\mathbf{g}(X)$
(0011)	(0100011)	$X + X^5 + X^6 = (X + X^2 + X^3)\mathbf{g}(X)$
(1011)	(1001011)	$1 + X^3 + X^5 + X^6 = (1 + X + X^2 + X^3)\mathbf{g}(X)$
(0111)	(0010111)	$X^2 + X^4 + X^5 + X^6 = (X^2 + X^3)\mathbf{g}(X)$
(1111)	(1111111)	$1 + X + X^2 + X^3 + X^4 + X^5 + X^6 = (1 + X^2 + X^5)\mathbf{g}(X)$

<u>Theorem 5</u>: The generator polynomial of an (n,k) code is a factor of $X^n + 1$.

<u>Proof</u>: Divide $X^k g(x)$ by $X^n + 1$

$$x^k g(x) = (x^n + 1) + g^{(k)}(x)$$

or

$$x^{n+1} = x^k g(x) + g^{(k)}(x)$$

$g^{(k)}(x)$ is a code polynomial. So $g^{(k)}(x) = a(x) g(x)$

for some $a(X)$. So,

$$X^n + 1 = [X^k + a(X)]\, g(X) \qquad \text{QED}$$

<u>Theorem 6</u>: If $g(x)$ is a polynomial of degree $n-k$ and is a factor of $X^n + 1$. Then $g(X)$ generates an (n,k) cyclic code.

Proof: Let $g(X), Xg(X), \cdots X^{k-1} g(X)$.

They are all polynomials of degree $n-1$ or less.

A linear combination of them:

$$v(X) = u_0\, g(X) + u_1 X g(X) + \cdots + u_{k-1} X^{k-1} g(X)$$

$$= [u_0 + u_1 X + \cdots + u_{k-1} X^{k-1}]\, g(X)$$

is a code polynomial since $u_i \in \{0,1\}$, then $v(X)$ will have 2^k possibilities.

These 2^k polynomials form the 2^k codewords of the (n,k) code.

Generator polynomial of a Cyclic Code:

$$
\mathbf{G} = \begin{bmatrix}
g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & 0 & 0 & \cdot & \cdot & 0 \\
0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & 0 & \cdot & \cdot & 0 \\
0 & 0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & \cdot & \cdot & 0 \\
\cdot & & & & & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & & & & & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & 0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k}
\end{bmatrix}.
$$

For example for (7,4) Code with

$g(x) = 1 + x + x^3$

$g_0 = g_1 = g_3 = 1$, $g_i = 0$ otherwise.

$$
G = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

This is not always in Systematic form. We can make it into systematic form by row and column operations. For example, for the (7,4) Code

$$
G' = \begin{bmatrix}
\underline{g}_0 \\
\underline{g}_1 \\
\underline{g}_0 + \underline{g}_2 \\
\underline{g}_0 + \underline{g}_1 + \underline{g}_2
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Parity check Matrix of Cyclic Codes

We saw that $g(x)$ divides x^n+1. Write

$$x^n+1 = g(x)h(x)$$

where $h(x)$ is a polynomial of degree k.

$$h(x) = h_0 + h_1 x + \dots + h_k x^k$$

Consider a code polynomial $v(x)$

$$v(x)h(x) = u(x)g(x)h(x)$$
$$= u(x)(x^n+1) = u(x)x^n + u(x)$$

Since $u(x)$ has degree less than or equal $k-1$

So $u(x) + x^n u(x)$ does not have $x^k, x^{k+1}, \dots$
$\dots x^{n-1}$. That is coeficients of these powers of x are zero. So we get $n-k$ equalities:

$$\sum_{i=0}^{k} h_i v_{n-i-j} = 0 \quad \text{for} \quad 1 \le j \le n-k.$$

So, we have H as:

$$\mathbf{H} = \begin{bmatrix} h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & \cdot & \cdot & 0 \\ \vdots & & & & & & & & & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & h_0 \end{bmatrix}.$$

Theorem 7: Let $g(x)$ be the generator polynomial of the (n,k) cyclic code C. The dual code of C is generated by $x^k h(x^{-1})$ where $h(x) = \frac{x^n+1}{g(x)}$.

Example: Consider $(7,4)$ code C with $g(x) = 1+x+x^3$

The generator polynomial of $C^\dagger$ is:

$$x^4 h(x^{-1}) \text{ where}$$

$$h(x) = \frac{x^7+1}{1+x+x^3} = 1+x+x^2+x^4$$

That is the generator of $C^\dagger$ is:

$$x^4 h(x^{-1}) = x^4(1+x^{-1}+x^{-2}+x^{-4})$$

$$= 1+x^2+x^3+x^4.$$

So $C^\dagger$ is a $(7,3)$ code with $d_{min} = 4$.

So, it can correct any single error and detect any combination of double errors.

Encoding of Cyclic Codes

we saw that if we multiply the information polynomial by X^{n-k} and divide by $g(x)$, we get:

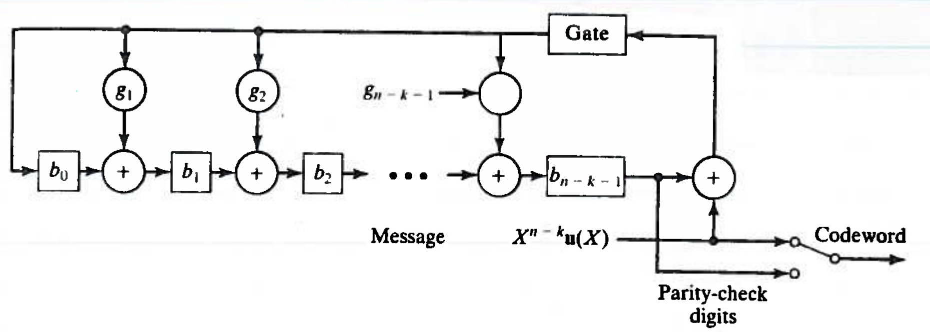
$$X^{n-1}\underline{u}(x) = a(x)\,g(x) + b(x)$$

and

$$a(x)\,g(x) = b(x) + X^{n-1}u(x)$$

is a Codeword in systematic form.

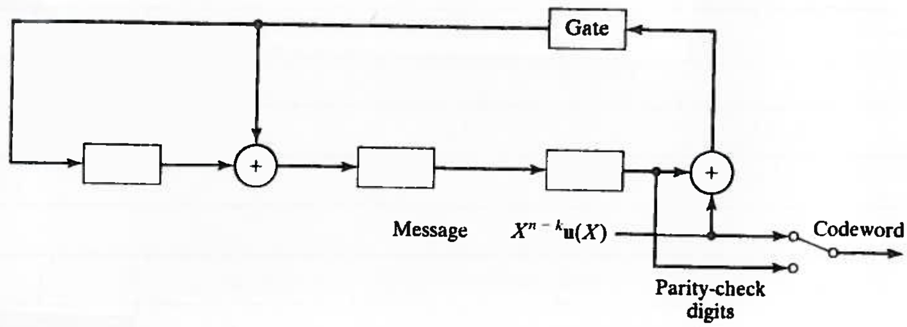
The following circuit encodes $u(x)$ based on the above discussion:



Encoding circuit for an (n, k) cyclic code with generator polynomial $\mathbf{g}(X) = 1 + g_1X^2 + \cdots + g_{n-k-1}X^{n-k-1} + X^{n-k}$.

1) Close the gate and enter information bits in and also send them over channel. This does multiplication by X^{n-k} as well as parity bit generation.
2) open the gate (break the feedback)
3) output the $n-k$ parity bits.

Example: (7,4) code with $g(X) = 1 + X + X^3$



Encoder for the (7, 4) cyclic code generated by $g(X) = 1 + X + X^3$.

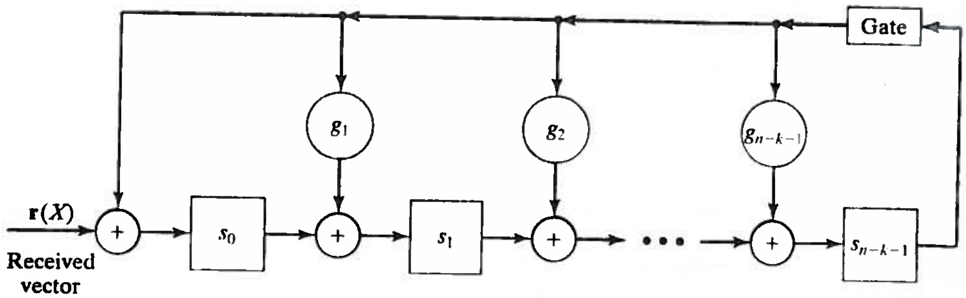
Syndrome

Assume $r(X) = r_0 + r_1 X + r_2 X^2 + \cdots + r_{n-1} X^{n-1}$ is the polynomial representing received bits. Divide $r(X)$ by $g(X)$ to get:

$$r(X) = a(X) g(X) + s(X)$$

$s(X)$ is a polynomial of degree $n-k-1$ or less. The $n-k$ coefficients of $s(X)$ are the syndromes.

Theorem 8: Let $s(X)$ be the syndrome of $r(X) = r_0 + r_1 X + \cdots r_{n-1} X^{n-1}$. Then $s^{(i)}(X)$ resulting from dividing $X^i s(X)$ by $g(X)$ is the syndrome of $r^{(i)}(X)$.



An $(n-k)$-stage syndrome circuit with input from the left end.

Example of (7,4) Code:

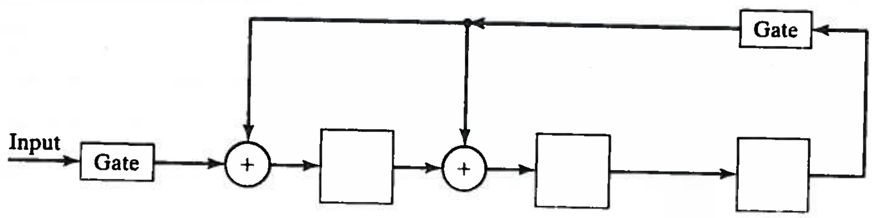
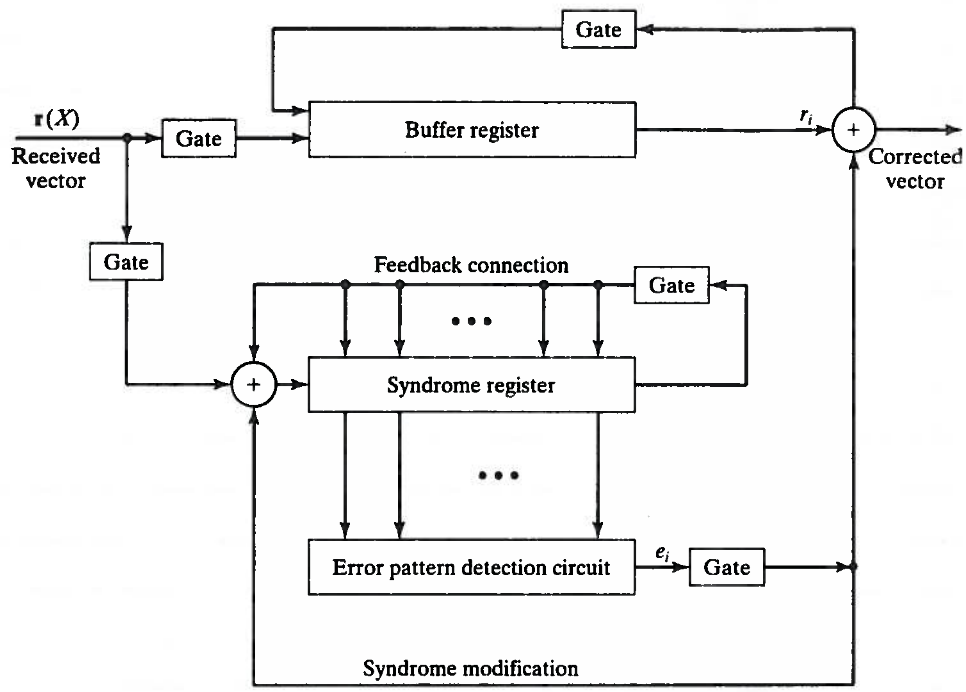


FIGURE 5.6: Syndrome circuit for the (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

TABLE 5.3: Contents of the syndrome register shown in Figure 5.6 with $\mathbf{r}$ = (0 0 1 0 1 1 0) as input.

Shift	Input	Register contents
		000 (initial state)
1	0	000
2	1	100
3	1	110
4	0	011
5	1	011
6	0	111
7	0	101 (syndrome $\mathbf{s}$)
8	—	100 (syndrome $\mathbf{s}^{(1)}$)
9	—	010 (syndrome $\mathbf{s}^{(2)}$)

Decoding:

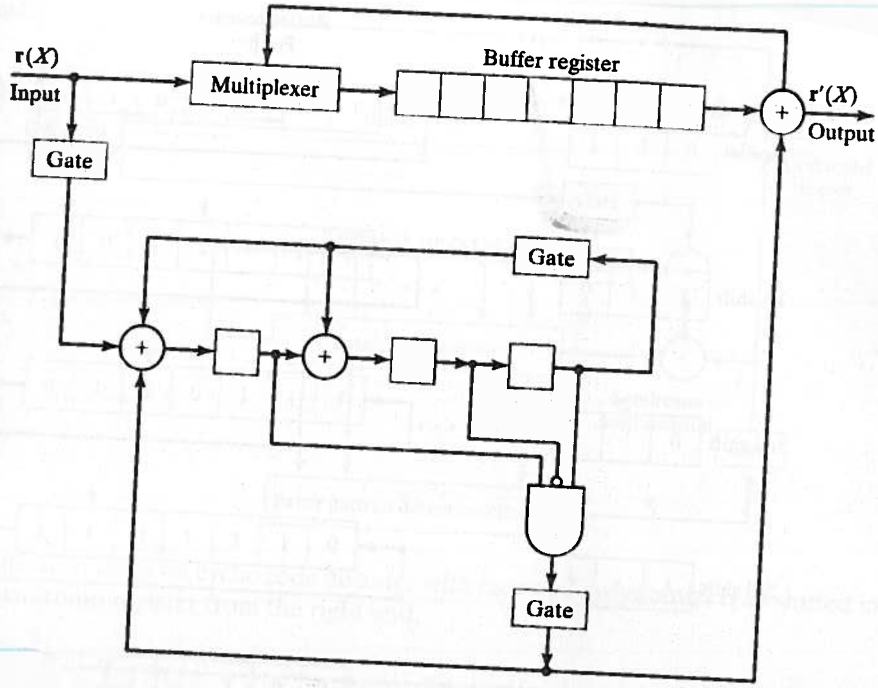


General cyclic code decoder with received polynomial $\mathbf{r}(X)$ shifted into the syndrome register from the left end.

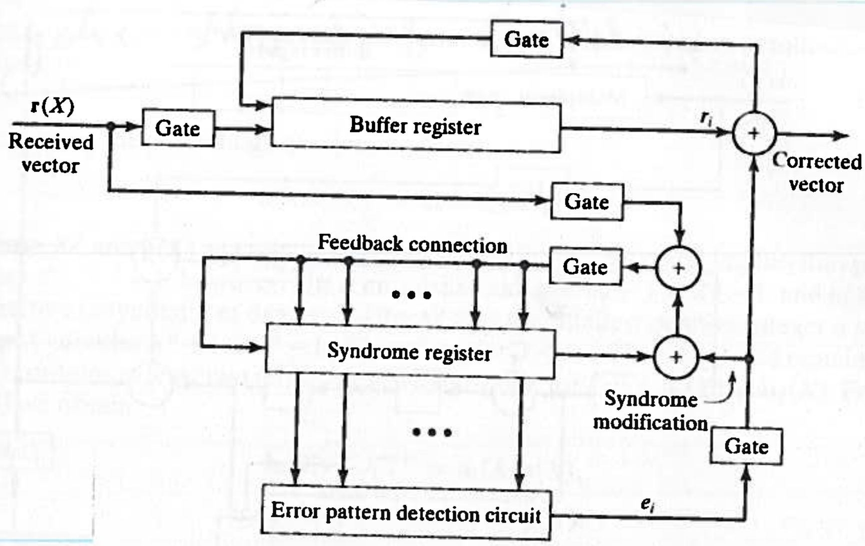
Example of (7,4) Code:

Error patterns and their syndromes with the received polynomial $\mathbf{r}(X)$ shifted into the syndrome register from the left end.

Error pattern $\mathbf{e}(X)$	**Syndrome** $\mathbf{s}(X)$	**Syndrome vector** (s_0, s_1, s_2)
$\mathbf{e}_6(X) = X^6$	$\mathbf{s}(X) = 1 + X^2$	(101)
$\mathbf{e}_5(X) = X^5$	$\mathbf{s}(X) = 1 + X + X^2$	(111)
$\mathbf{e}_4(X) = X^4$	$\mathbf{s}(X) = X + X^2$	(011)
$\mathbf{e}_3(X) = X^3$	$\mathbf{s}(X) = 1 + X$	(110)
$\mathbf{e}_2(X) = X^2$	$\mathbf{s}(X) = X^2$	(001)
$\mathbf{e}_1(X) = X^1$	$\mathbf{s}(X) = X$	(010)
$\mathbf{e}_0(X) = X^0$	$\mathbf{s}(X) = 1$	(100)



Decoding circuit for the (7,4) cyclic code generated by $g(X) = 1+X+X^3$.



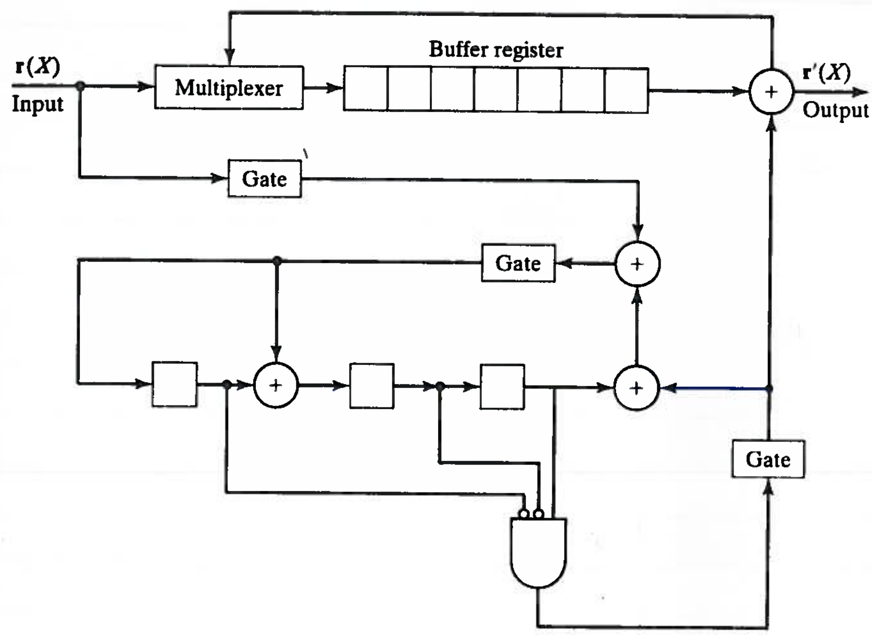
General cyclic code decoder with received polynomial r(X) shifted into the syndrome register from the right end.

Another implementation of Syndrome calculator.

Syndrom Decoding of (7,4) Code using Syndrome Decoder fed from right.

Error patterns and their syndromes with the received polynomial $r(X)$ shifted into the syndrome register from the right end.

Error pattern $e(X)$	Syndrome $s^{(3)}(X)$	Syndrome vector (s_0, s_1, s_2)
$e(X) = X^6$	$s^{(3)}(X) = X^2$	(001)
$e(X) = X^5$	$s^{(3)}(X) = X$	(010)
$e(X) = X^4$	$s^{(3)}(X) = 1$	(100)
$e(X) = X^3$	$s^{(3)}(X) = 1 + X^2$	(101)
$e(X) = X^2$	$s^{(3)}(X) = 1 + X + X^2$	(111)
$e(X) = X$	$s^{(3)}(X) = X + X^2$	(011)
$e(X) = X^0$	$s^{(3)}(X) = 1 + X$	(110)



Decoding circuit for the (7, 4) cyclic code generated by $g(X) = 1 + X + X^3$.

Cyclic Hamming Codes:

A Hamming Code of length $n = 2^m - 1$ with $m \geq 3$ is generated by a primitive polynomial of degree m.

Let's see how we can put the Hamming code with defined in last lecture in cyclic form:

Divide X^{m+i} by $p(x)$ to get

$$X^{m+i} = a_i(x)p(x) + b_i(x)$$

1) Since $p(x)$ is primitive, X is not a factor of $p(X)$ so $p(x)$ does not divide X^{m+i} $\Rightarrow$ $\underline{\underline{b_i(X) \neq 0}}$

2) $b_i(X)$ has at least two terms

if it had one term:

$$X^{m+i} = a_i(X)p(X) + X^j$$

$$\Rightarrow X^j(X^{m+i-j} + 1) = a_i(X)p(x)$$

$\Rightarrow p(X)$ divides $X^{m+i-j}+1$ but $m+i-j < 2^m - 1$

$\Rightarrow$ Contradiction.

3) if $i \neq j$ then $b_i(X) \neq b_j(X)$

Let

$$X^{m+i} = b_i(X) + a_i(X)p(X)$$

$$X^{m+j} = b_j(X) + a_j(X)p(X)$$

if $b_i(X) = b_j(X)$ then

$$X^{m+i}(X^{j-i}+1) = [a_i(X)+a_j(X)]\,p(X)$$

i.e., $p(X)$ divides $X^{j-i}+1 \Rightarrow$ Contradiction.

Let $H = [I_m : Q]$ be the parity check matrix of this Code. I_m is an $m \times m$ identity matrix with Q an $m \times (2^m - m - 1)$ matrix with $\underline{b}_i = (b_{i0}, b_{i1}, \ldots b_{i,m-1})$ as its columns. Since no two columns of Q are the same and each have at least two 1's, then H is indeed a parity-check matrix of a Hamming Code.

Syndrome Decoding of Hamming Codes

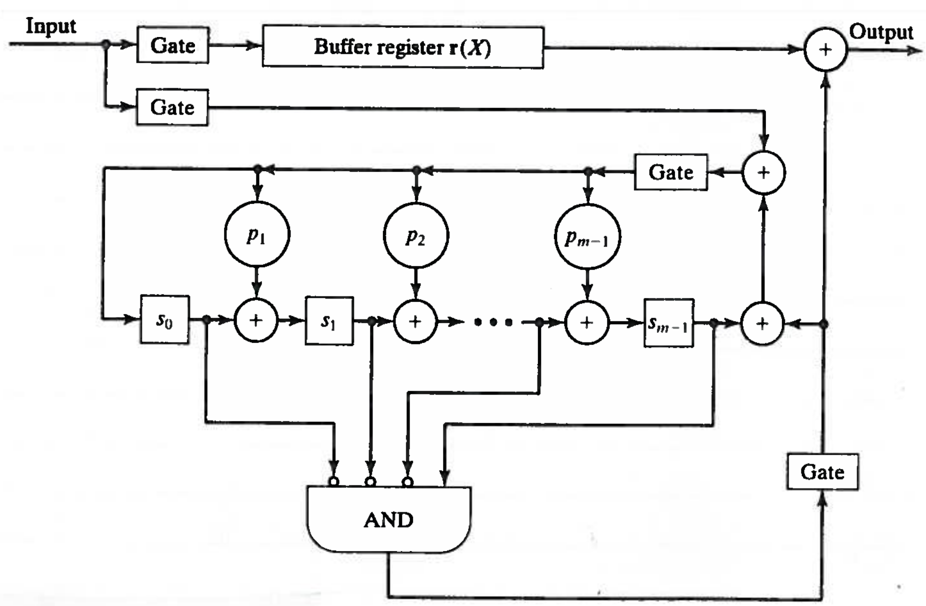
Assume that error is in location with highest order, i.e.,

$$e(X) = X^{2^m - 2}$$

then feeding $r(X)$ from right to syndrome Calculator is equivalent to dividing $X^m \cdot X^{2^m - 2}$ by the generator polynomial $p(X)$. Since $p(X)$ divides $X^{2^m-1}+1$ then

$$s(X) = X^{m-1}$$

or $\underline{s} = (0, 0, \ldots, 0, 1)$



Decoder for a cyclic Hamming code.