$[\mathbf{v}]_l = (1\,1, 0\,1, 0\,0, 0\,0, \cdots, 0\,0)$, even in the limit as $l \to \infty$. Note, however, that all finite-length paths in the state diagram that diverge from and remerge with the all-zero state $S_0$ have a weight of at least 4, and hence, $d_{free} = 4$. In this case we have a situation in which $\lim_{l\to\infty} d_l = 3 \neq d_{free} = 4$; that is, (11.168) is not satisfied.

It is characteristic of catastrophic encoders that an infinite-weight information sequence produces a finite-weight codeword. In some cases, as in the preceding example, this codeword can have a weight less than the free distance of the code, owing to the zero-output weight cycle in the state diagram. In other words, an information sequence that traverses this zero-output weight cycle forever will itself pick up infinite weight without adding to the weight of the codeword. In a noncatastrophic encoder, which contains no zero-output weight cycle other than the zero-weight cycle around the state $S_0$, all infinite-weight information sequences must generate infinite-weight codewords, and the minimum weight codeword always has finite length. Unfortunately, the information sequence that produces the minimum-weight codeword may be quite long in some cases, and hence the calculation of $d_{free}$ can be a difficult task.

The best achievable $d_{free}$ for a convolutional code with a given rate $R$ and overall constraint length $\nu$ has not been determined in general; however, upper and lower bounds on $d_{free}$ have been obtained using a random coding approach. These bounds are thoroughly discussed in References [16], [17], and [18]. A comparison of the bounds for nonsystematic encoders with the bounds for systematic encoders implies that more free distance is available with nonsystematic feedforward encoders of a given rate and constraint length than with systematic feedforward encoders. This observation is verified by the code construction results presented in the next two chapters and has important consequences when a code with large $d_{free}$ must be selected for use with ML, MAP, or sequential decoding. Thus, if a systematic encoder realization is desired, it is usually better to select a nonsystematic feedforward encoder with large $d_{free}$ and then convert it to an equivalent systematic feedback encoder.

## PROBLEMS

✱ **11.1** Consider the (3, 1, 2) nonsystematic feedforward encoder with

$$\mathbf{g}^{(0)} = (1\,1\,0),$$
$$\mathbf{g}^{(1)} = (1\,0\,1),$$
$$\mathbf{g}^{(2)} = (1\,1\,1).$$

**a.** Draw the encoder block diagram.
**b.** Find the time-domain generator matrix **G**.
**c.** Find the codeword **v** corresponding to the information sequence $\mathbf{u} = (1\,1\,1\,0\,1)$.

✱ **11.2** Consider the (4, 3, 3) nonsystematic feedforward encoder shown in Figure 11.3.
**a.** Find the generator sequences of this encoder.
**b.** Find the time-domain generator matrix **G**.
**c.** Find the codeword **v** corresponding to the information sequence $\mathbf{u} = (110, 011, 101)$.

*11.3 Consider the (3, 1, 2) encoder of Problem 11.1.
   **a.** Find the transform-domain generator matrix $\mathbf{G}(D)$.
   **b.** Find the set of output sequences $\mathbf{V}(D)$ and the codeword $\mathbf{v}(D)$ corresponding to the information sequence $\mathbf{u}(D) = 1 + D^2 + D^3 + D^4$.

**11.4** Consider the (3, 2, 2) nonsystematic feedforward encoder shown in Figure 11.2.
   **a.** Find the composite generator polynomials $\mathbf{g}_1(D)$ and $\mathbf{g}_2(D)$.
   **b.** Find the codeword $\mathbf{v}(D)$ corresponding to the set of information sequences $\mathbf{U}(D) = [1 + D + D^3 \quad 1 + D^2 + D^3]$.

* **11.5** Consider the (3, 1, 5) systematic feedforward encoder with

$$\mathbf{g}^{(1)} = (1\,0\,1\,1\,0\,1),$$
$$\mathbf{g}^{(2)} = (1\,1\,0\,0\,1\,1).$$

   **a.** Find the time-domain generator matrix $\mathbf{G}$.
   **b.** Find the parity sequences $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ corresponding to the information sequence $\mathbf{u} = (1\,1\,0\,1)$.

**11.6** Consider the (3, 2, 3) systematic feedforward encoder with

$$\mathbf{g}_1^{(2)}(D) = 1 + D^2 + D^3,$$
$$\mathbf{g}_2^{(2)}(D) = 1 + D + D^3.$$

   **a.** Draw the controller canonical form realization of this encoder. How many delay elements are required in this realization?
   **b.** Draw the simpler observer canonical form realization that requires only three delay elements.

**11.7** Verify the sequence of elementary row operations leading from the nonsystematic feedforward realizations of (11.34) and (11.70) to the systematic feedback realizations of (11.66) and (11.71).

**11.8** Draw the observer canonical form realization of the generator matrix $\mathbf{G}'(D)$ in (11.64) and determine its overall constraint length $\nu$.

**11.9** Consider the rate $R = 2/3$ nonsystematic feedforward encoder with generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} D & D & 1 \\ 1 & D^2 & 1 + D + D^2 \end{bmatrix}.$$

   **a.** Draw the controller canonical form encoder realization for $\mathbf{G}(D)$. What is the overall constraint length $\nu$?
   **b.** Find the generator matrix $\mathbf{G}'(D)$ of the equivalent systematic feedback encoder. Is $\mathbf{G}'(D)$ realizable? If not, find an equivalent realizable generator matrix and draw the corresponding minimal encoder realization. Is this minimal realization in controller canonical form or observer canonical form? What is the minimal overall constraint length $\nu$?

**11.10** Use elementary row operations to convert the rate $R = 2/3$ generator matrix of (11.77) to systematic feedback form, and draw the minimal observer canonical form encoder realization. Find and draw a nonsystematic feedback controller canonical form encoder realization with the same number of states.

**11.11** Redraw the observer canonical form realization of the (3, 2, 2) systematic feedback encoder in Figure 11.7(b) using the notation of (11.82) and the relabeling scheme of Figure 11.11.

**11.12** Consider the $(3, 1, 2)$ systematic feedback encoder shown in Figure 11.6(c). Determine the $\nu = 2$ termination bits required to return this encoder to the all-zero state when the information sequence $\mathbf{u} = (10111)$.

**11.13** Consider the $(4, 3, 3)$ nonsystematic feedforward encoder realization in controller canonical form shown in Figure 11.3.

**a.** Draw the equivalent nonsystematic feedforward encoder realization in observer canonical form, and determine the number of termination bits required to return this encoder to the all-zero state. What is the overall constraint length of this encoder realization?

**b.** Now, determine the equivalent systematic feedback encoder realization in observer canonical form, and find the number of termination bits required to return this encoder to the all-zero state. What is the overall constraint length of this encoder realization?

**11.14** Consider the $(2, 1, 2)$ nonsystematic feedforward encoder with $\mathbf{G}(D) = [1 + D^2 \;\; 1 + D + D^2]$.

**a.** Find the GCD of its generator polynomials.

**b.** Find the transfer function matrix $\mathbf{G}^{-1}(D)$ of its minimum-delay feedforward inverse.

**11.15** Consider the $(2, 1, 3)$ nonsystematic feedforward encoder with $\mathbf{G}(D) = [1 + D^2 \;\; 1 + D + D^2 + D^3]$.

**a.** Find the GCD of its generator polynomials.

**b.** Draw the encoder state diagram.

**c.** Find a zero-output weight cycle in the state diagram.

**d.** Find an infinite-weight information sequence that generates a codeword of finite weight.

**e.** Is this encoder catastrophic or noncatastrophic?

**11.16** Find the general form of transfer function matrix $\mathbf{G}^{-1}(D)$ for the feedforward inverse of an $(n, k, \nu)$ systematic encoder. What is the minimum delay $l$?

**11.17** Verify the calculation of the WEF in Example 11.13.

**11.18** Verify the calculation of the IOWEF in Example 11.12.

**11.19** Consider the $(3, 1, 2)$ encoder of Problem 11.1.

**a.** Draw the state diagram of the encoder.

**b.** Draw the modified state diagram of the encoder.

**c.** Find the WEF $A(X)$.

**d.** Draw the augmented modified state diagram of the encoder.

**e.** Find the IOWEF $A(W, X, L)$.

**11.20** Using an appropriate software package, repeat Problem 11.18 for the $(4, 3, 3)$ encoder of Figure 11.3.

**11.21** Consider the equivalent systematic feedback encoder for Example 11.1 obtained by dividing each generator polynomial by $\mathbf{g}^{(0)}(D) = 1 + D^2 + D^3$.

**a.** Draw the augmented modified state diagram for this encoder.

**b.** Find the IRWEF $A(W, Z)$, the two lowest input weight CWEFs, and the WEF $A(X)$ for this encoder.

**c.** Compare the results obtained in (b) with the IOWEF, CWEFs, and WEF computed for the equivalent nonsystematic feedforward encoder in Example 11.1.

**11.22** Verify the calculation of the IOWEF given in (11.124) for the case of a terminated convolutional encoder.

**11.23** Consider the equivalent nonsystematic feedforward encoder for Example 11.14 obtained by multiplying $\mathbf{G}(D)$ in (11.140) by $\mathbf{g}^{(0)}(D) = 1 + D + D^2$.

a. Draw the augmented modified state diagram for this encoder.
b. Find the IOWEF $A(W, X, L)$, the three lowest input weight CWEFs, and the WEF $A(X)$ for this encoder.
c. Compare the results obtained in (b) with the IRWEF, CWEFs, and WEF computed for the equivalent systematic feedback encoder in Example 11.14.

**11.24** In Example 11.14, verify all steps leading to the calculation of the bit WEF in (11.154).

**11.25** Consider the (2, 1, 2) systematic feedforward encoder with $\mathbf{G}(D) = [1 \;\; 1 + D^2]$.
a. Draw the augmented modified state diagram for this encoder.
b. Find the IRWEF $A(W, Z, L)$, the three lowest input weight CWEFs, and the WEF $A(X)$ for this encoder.

**11.26** Recalculate the IOWEF $A(W, X, L)$ in Example 11.12 using the state variable approach of Example 11.14.

**11.27** Recalculate the WEF $A(X)$ in Example 11.13 using the state variable approach of Example 11.14.

**11.28** Consider the (3, 1, 2) code generated by the encoder of Problem 11.1.
a. Find the free distance $d_{free}$.
b. Plot the complete CDF.
c. Find the minimum distance $d_{min}$.

**11.29** Repeat Problem 11.28 for the code generated by the encoder of Problem 11.15.

**11.30** a. Prove that the free distance $d_{free}$ is independent of the encoder realization, i.e., it is a code property.
b. Prove that the CDF $d_l$ is independent of the encoder realization; that is, it is a code property. (Assume that the $k \times n$ submatrix $\mathbf{G}_0$ has full rank.)

**11.31** Prove that for noncatastrophic encoders

$$\lim_{l \to \infty} d_l = d_{free}.$$

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