ELEC 6131: Error Detecting and Correcting Codes

Instructor:
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Time and Place: Tuesday, 17:45 – 20:15.
Office Hours: Tuesday, 15:00 – 17:00

LECTURE 5: Cyclic Codes

Outline of this lecture

- In this lecture we cover the following:
  - Brief discussion of Hamming codes,
  - Cyclic Codes.

Hamming Codes

- Code length: $n = 2^m - 1$
- # of information bits: $k = 2^m - 1 - m$ and # of parity bits: $n - k = m$

$$d_{min} = 3 \;\Rightarrow\; t = 1$$

- The parity check matrix of this code H contains all m-tuples except $00 \cdots 0$ as its columns. They are arranged to look like:

$$H = [I_m \colon Q].$$

For example, for $m = 3$, we have

$$H = \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \;\vdots\; \begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{matrix}\right].$$

and $G = [Q^T \colon I_{2^m - m - 1}]$.

Hamming Codes

- Since H consists all the m-tuples as its columns, adding any two columns, we get another column, i.e.,

$$\underline{h}_i + \underline{h}_j + \underline{h}_k = 0.$$

- So, the minimum distance of the code is not greater than 3. Also, since we do not have any two columns that add up to $\underline{0}$, the minimum distance of the code is not less than 3. Therefore, $d_{min} = 3$.

- Hamming codes are perfect codes: if we form standard array, it will contain $2^n = 2^{2^m-1}$ elements. Each row has $2^k = 2^{2^m-m-1}$ elements. So, there will be $\frac{2^{2^m-1}}{2^{2^m-m-1}} = 2^m$ cosets. Therefore, in addition to $\underline{0}$ we need $2^m - 1$ coset leaders. If we take all single error patterns, we have exactly what we need. So, a Hamming code only corrects error patterns with one erroneous bit and corrects all of these. So, Hamming codes are perfect codes.
- The only other binary perfect code is $(23, 12)$ Golay code.

Hamming Codes

- Weight distribution: let A_i be the number of codewords of weight i. Then, $A(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$ can be formed. It is called weight enumerator. For a Hamming code:

$$A(z) = \frac{1}{n+1}\left[(1+z)^n + n(1-z)(1-z^2)^{\frac{n-1}{2}}\right].$$

- **Example:** Consider $m = 3$.

$$n = 2^m - 1 = 2^3 - 1 = 7 \quad \Rightarrow \quad (7,4) \text{ code}$$

$$A(z) = \frac{1}{8}[(1+z)^7 + 7(1-z)(1-z^2)^3]$$

$$= 1 + 7z^3 + 7z^4 + z^7.$$

Cyclic Codes

- **Definition:** a linear block code is <u>cyclic</u> if a cyclic shift of any codeword is another codeword.

The ith shift of $\underline{v} = (v_0, v_1, \cdots, v_{n-1})$ is:

$$\underline{v}^{(i)} = (v_{n-i}, v_{n-i+1}, \cdots, v_{n-1}, v_0, v_1, \cdots, v_{n-i-1}).$$

- For example, $\underline{v}^{(1)} = (v_{n-1}, v_0, v_1, \cdots, v_{n-2})$ and $\underline{v}^{(2)} = (v_{n-2}, v_{n-1}, v_0, v_1, \cdots, v_{n-3})$.

- **Example:** (7, 4) Hamming Code (see next slide).

Cyclic Codes

A (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

Messages	Code vectors	Code polynomials
(0000)	0000000	$0 = 0 \cdot \mathbf{g}(X)$
(1000)	1101000	$1 + X + X^3 = 1 \cdot \mathbf{g}(X)$
(0100)	0110100	$X + X^2 + X^4 = X \cdot \mathbf{g}(X)$
(1100)	1011100	$1 + X^2 + X^3 + X^4 = (1 + X) \cdot \mathbf{g}(X)$
(0010)	0011010	$X^2 + X^3 + X^5 = X^2 \cdot \mathbf{g}(X)$
(1010)	1110010	$1 + X + X^2 + X^5 = (1 + X^2) \cdot \mathbf{g}(X)$
(0110)	0101110	$X + X^3 + X^4 + X^5 = (X + X^2) \cdot \mathbf{g}(X)$
(1110)	1000110	$1 + X^4 + X^5 = (1 + X + X^2) \cdot \mathbf{g}(X)$
(0001)	0001101	$X^3 + X^4 + X^6 = X^3 \cdot \mathbf{g}(X)$
(1001)	1100101	$1 + X + X^4 + X^6 = (1 + X^3) \cdot \mathbf{g}(X)$
(0101)	0111001	$X + X^2 + X^3 + X^6 = (X + X^3) \cdot \mathbf{g}(X)$
(1101)	1010001	$1 + X^2 + X^6 = (1 + X + X^3) \cdot \mathbf{g}(X)$
(0011)	0010111	$X^2 + X^4 + X^5 + X^6 = (X^2 + X^3) \cdot \mathbf{g}(X)$
(1011)	1111111	$1 + X + X^2 + X^3 + X^4 + X^5 + X^6$ $= (1 + X^2 + X^3) \cdot \mathbf{g}(X)$
(0111)	0100011	$X + X^5 + X^6 = (X + X^2 + X^3) \cdot \mathbf{g}(X)$
(1111)	1001011	$1 + X^3 + X^5 + X^6$ $= (1 + X + X^2 + X^3) \cdot \mathbf{g}(X)$

Cyclic Codes

- Let $v(X) = v_0 + v_1 X + v_2 X^2 + \cdots + v_{n-1} X^{n-1}$ be the polynomial representation of $\underline{v}$.
- Then $v^{(i)}(X) = v_{n-i} + v_{n-i+1} X + \cdots + v_{n-1} X^{i-1} + v_0 X^i + v_1 X^{i+1} + \cdots + v_{n-i-1} X^{n-1}$.

Multiply X^i by $v(X)$, i.e., shift $\underline{v}$ i times (linearly, not cyclically) to get:

$$X^i v(X) = v_0 X^i + v_1 X^{i+1} + \cdots + v_{n-i+1} X^{n-1} + \cdots + v_{n-1} X^{n+i-1}.$$

Add $X^i v(X)$ and $v^{(i)}(X)$:

$$X^i v(X) + v^{(i)}(X)$$

$$= v_{n-i} + v_{n-i+1} X + \cdots + v_{n-1} X^{i-1} + v_{n-i} X^n + v_{n-i+1} X^{n+1} + \cdots + v_{n-1} X^{n+i-1}$$

or:

$$X^i v(X) + v^{(i)}(X) = [v_{n-i} + v_{n-i+1} X + \cdots + v_{n-1} X^{i-1}](X^n + 1).$$

So:

$$X^i v(X) = q(X)[X^n + 1] + v^{(i)}(X).$$

That is, the ith cyclic shift of $v(X)$ is generated by dividing $X^i v(X)$ by $X^n + 1$.

Cyclic Codes

- **Theorem 1:** the non-zero code polynomial with minimum degree in a cyclic code C is unique.

Proof: let $g(X) = g_0 + g_1X + \cdots + g_{r-1}X^{r-1} + X^r$ be the minimal degree code polynomial of C. Suppose there is another $g'(X) = g'_0 + g'_1X + \cdots + g'_{r-1}X^{r-1} + X^r$. Then, $g(X) + g'(X)$ is another codeword in C with degree less than r. $\Rightarrow$ contradiction.

- **Theorem 2:** let $g(X) = g_0 + g_1X + \cdots + g_{r-1}X^{r-1} + X^r$ be the minimum degree polynomial of a cyclic code C. Then, $g_0 \neq 0$.

Proof: if $g_0 = 0$ then shifting $g(X)$ once to the left (or $n-1$ times to right) results in $g_1 + g_2X + \cdots + g_{r-1}X^{r-2} + X^{r-1}$ which has a degree $< r$ $\Rightarrow$ contradiction. So, $g(X) = 1 + g_1X + \cdots + g_{r-1}X^{r-1} + X^r$.

- Let $g(X)$ be the polynomial of minimum degree of a code C. Take $g(X), Xg(X), X^2g(X), \cdots, X^{n-r-1}g(X)$. These are shifts of $g(X)$ by $0, 1, \cdots, n-r-1$. So, they are codewords. Any linear combination of them is also a codeword. Therefore,

$$v(X) = u_0g(X) + u_1Xg(X) + \cdots + u_{n-r-1}X^{n-r-1}g(X)$$

$$= [u_0 + u_1X + \cdots + u_{n-r-1}X^{n-r-1}]g(X)$$

is also a codeword.

Cyclic Codes

- **Theorem 3:** let $g(X) = 1 + g_1 X + \cdots + g_{r-1} X^{r-1} + X^r$ be the non-zero code polynomial of minimum degree of an (n, k) cyclic code C. A binary polynomial of degree $n-1$ or less is a code polynomial if and only if it is a multiple of $g(X)$.

Proof: let $v(X)$ be a polynomial of degree $n-1$ or less such that:

$$v(X) = (a_0 + a_1 X + \cdots + a_{n-r-1} X^{n-r-1}) g(X).$$

Then,

$$v(X) = a_0 g(X) + a_1 X g(X) + \cdots + a_{n-r-1} X^{n-r-1} g(X).$$

Since $g(X), Xg(X), \cdots$ are each codeword of C so is their sum $v(X)$.

Now assume $v(X)$ be a code polynomial in C. Then write:

$$v(X) = a(X)g(X) + b(X)$$

i.e., divide $v(X)$ by $g(X)$ and get remainder $b(X)$ and quotient $a(X)$.

$$b(X) = v(X) + a(X)g(X).$$

$v(X)$ is a codeword and so is $a(X)g(X)$. Therefore, $b(X)$ is also a codeword. But degree of $b(X)$ is less than $r \Rightarrow$ contradiction unless if $b(X) = 0$.

- The number of polynomials of degree $n-1$ or less that are multiple of $g(X)$ is 2^{n-r}. Due to 1-to-1 correspondence between these polynomials and the codewords (Theorem 3), we have $2^{n-r} = 2^k \Rightarrow r = n - k$.

Cyclic Codes

- **Theorem 4:** in an (n, k) cyclic code, there is one and only one code polynomial of degree $n-k$,

$$g(X) = 1 + g_1X + g_2X^2 + \cdots + g_{n-k-1}X^{n-k-1} + X^{n-k}.$$

- Every code polynomial is a multiple of $g(X)$. Every binary polynomial of degree $n-1$ or less that is a multiple of $g(X)$ is a code polynomial. So,

$$v(X) = u(X)g(X)$$

is a code polynomial, however, not in a systematic form.

- To make a cyclic code systematic, multiply the information polynomial $u(X)$ by X^{n-k}. This means placing the k information bits at the head of the shift register (in k right-most Flip-Flops). Then,

$$u(X) = u_0 + u_1X + \cdots + u_{k-1}X^{k-1}$$

will result in:

$$X^{n-k}u(X) = u_0X^{n-k} + u_1X^{n-k+1} + \cdots + u_{k-1}X^{n-1}.$$

Cyclic Codes

- Now divide $X^{n-k}u(X)$ by $g(X)$ to get:

$$X^{n-k}u(X) = a(X)g(X) + b(X),$$

where $b(X)$ is a polynomial of degree $n-k-1$ or less:

$$b(X) = b_0 + b_1X + \cdots + b_{n-k-1}X^{n-k-1}$$

$$b(X) + X^{n-k}u(X) = a(X)g(X).$$

This means that $b(X) + X^{n-k}u(X)$ is the representation of a codeword in systematic form, i.e.,

$$b(X) + X^{n-k}u(X) = b_0 + b_1X + \cdots + b_{n-k-1}X^{n-k-1}$$

$$+ u_0X^{n-k} + u_1X^{n-k+1} + \cdots + u_{k-1}X^{n-1}$$

that represents

$$\underline{v} = (b_0, b_1, \cdots, b_{n-k-1}, u_0, u_1, \cdots, u_{k-1}).$$

Cyclic Codes

- **Example:** consider the $(7,4)$ cyclic code generated by $g(X) = 1 + X + X^3$. Let $u(X) = 1 + X^3$. Then,

1. $X^3 u(X) = X^3 + X^6$
2. Divide by $g(X) = 1 + X + X^3$

$$\begin{array}{r} X^3 + X \\ X^3 + X + 1 \;\overline{)\, X^6 + X^3} \\ X^6 + X^4 + X^3 \\ \hline X^4 \\ X^4 + X^2 + X \\ \hline X^2 + X \end{array}$$

$X^2 + X \leftarrow b(X)$

3. $v(X) = b(X) + X^3 u(X) =$

$X + X^2 + X^3 + X^6$

or $\underline{v} = (0,1,1,1,0,0,1)$

A (7, 4) cyclic code in systematic form generated by $\mathbf{g}(X) = 1 + X + X^3$.

Message	Codeword	
(0000)	(0000000)	$0 = 0 \cdot \mathbf{g}(X)$
(1000)	(1101000)	$1 + X + X^3 = \mathbf{g}(X)$
(0100)	(0110100)	$X + X^2 + X^4 = X\mathbf{g}(X)$
(1100)	(1011100)	$1 + X^2 + X^3 + X^4 = (1 + X)\mathbf{g}(X)$
(0010)	(1110010)	$1 + X + X^2 + X^5 = (1 + X^2)\mathbf{g}(X)$
(1010)	(0011010)	$X^2 + X^3 + X^5 = X^2\mathbf{g}(X)$
(0110)	(1000110)	$1 + X^4 + X^5 = (1 + X + X^2)\mathbf{g}(X)$
(1110)	(0101110)	$X + X^3 + X^4 + X^5 = (X + X^2)\mathbf{g}(X)$
(0001)	(1010001)	$1 + X^2 + X^6 = (1 + X + X^3)\mathbf{g}(X)$
(1001)	(0111001)	$X + X^2 + X^3 + X^6 = (X + X^3)\mathbf{g}(X)$
(0101)	(1100101)	$1 + X + X^4 + X^6 = (1 + X^3)\mathbf{g}(X)$
(1101)	(0001101)	$X^3 + X^4 + X^6 = X^3\mathbf{g}(X)$
(0011)	(0100011)	$X + X^5 + X^6 = (X + X^2 + X^3)\mathbf{g}(X)$
(1011)	(1001011)	$1 + X^3 + X^5 + X^6 = (1 + X + X^2 + X^3)\mathbf{g}(X)$
(0111)	(0010111)	$X^2 + X^4 + X^5 + X^6 = (X^2 + X^3)\mathbf{g}(X)$
(1111)	(1111111)	$1 + X + X^2 + X^3 + X^4 + X^5 + X^6$

Cyclic Codes

- **Theorem 5:** the generator polynomial of an (n, k) code is a factor of $X^n + 1$.

Proof: divide $X^k g(X)$ by $X^n + 1$.

$$X^k g(X) = (X^n + 1) + g^{(k)}(X) \text{ or } X^n + 1 = X^k g(X) + g^{(k)}(X)$$

$g^{(k)}(X)$ is a code polynomial. So, $g^{(k)}(X) = a(X)b(X)$ for some $a(X)$. So,

$$X^n + 1 = [X^k + a(X)]g(X). \qquad QED$$

- **Theorem 6:** if $g(X)$ is a polynomial of degree $n - k$ and is a factor of $X^n + 1$. Then $g(X)$ generates an (n, k) cyclic code.

Proof: let $g(X), Xg(X), \cdots, X^{k-1}g(X)$. They are all polynomials of degree $n - 1$ or less. A linear combination of them:

$$v(X) = u_0 g(X) + u_1 X g(X) + \cdots + u_{k-1} X^{k-1} g(X)$$
$$= [u_0 + u_1 X + \cdots + u_{k-1} X^{k-1}] g(X)$$

is a code polynomial since $u_i \in \{0, 1\}$. Then $v(X)$ will have 2^k possibilities. These 2^k polynomials form the 2^k codewords of the (n, k) code.

Cyclic Codes

Generator polynomial of a cyclic code:

$$\mathbf{G} = \begin{bmatrix} g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} & 0 & \cdot & \cdot & 0 \\ \cdot & & & & & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & & & & & \cdot \\ \cdot & & & & & & & & & & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & g_0 & g_1 & g_2 & \cdot & \cdot & \cdot & \cdot & \cdot & g_{n-k} \end{bmatrix}$$

For example, for $(7,4)$ code with $g(X) = 1 + X + X^3$, $g_0 = g_1 = g_3 = 1$ and $g_i = 0$ otherwise.

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Cyclic Codes

- This is not always in systematic form. We can make it into systematic form by row and column operations. For example, for the $(7, 4)$ code:

$$G' = \begin{bmatrix} \underline{g}_0 \\ \underline{g}_1 \\ \underline{g}_0 + \underline{g}_2 \\ \underline{g}_0 + \underline{g}_1 + \underline{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- **Parity check matrix of cyclic codes:**

We saw that $g(X)$ divides $X^n + 1$. Write

$$X^n + 1 = g(X)h(X),$$

where $h(X)$ is a polynomial of degree k

$$h(X) = h_0 + h_1 X + \cdots + h_k X^k.$$

Cyclic Codes

Consider a code polynomial $v(X)$

$$\begin{aligned} v(X)h(X) &= u(X)g(X)h(X) \\ &= u(X)(X^n + 1) \\ &= u(X)X^n + u(X). \end{aligned}$$

- Since $u(X)$ has degree less than or equal $k-1$, $u(X)X^n + u(X)$ does not have $X^k, X^{k+1}, \cdots, X^{n-1}$. That is coefficients of these powers of X are zero. So, we get $n-k$ equalities:

$$\sum_{i=0}^{k} h_i v_{n-i-j} = 0 \quad \text{for} \quad 1 \le j \le n-k.$$

- and we have H as:

$$\mathbf{H} = \begin{bmatrix} h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & & & & & & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & h_k & h_{k-1} & h_{k-2} & \cdot & \cdot & \cdot & \cdot & \cdot & h_0 \end{bmatrix}$$

Cyclic Codes

- **Theorem 7:** let $g(X)$ be the generator polynomial of the (n, k) cyclic code C.

  The dual code of C is generated by $X^k h(X^{-1})$ where $h(X) = \frac{X^n+1}{g(X)}$.
- **Example:** consider $(7, 4)$ code C with $g(X) = 1 + X + X^3$. The generator polynomial of C^d is $X^4 h(X^{-1})$ where,

$$h(X) = \frac{X^7 + 1}{1 + X + X^3} = 1 + X + X^2 + X^4.$$

That is, the generator of C^d is:

$$X^4 h(X^{-1}) = X^4(1 + X^{-1} + X^{-2} + X^{-4})$$

$$= 1 + X^2 + X^3 + X^4.$$

- So, C^d is a $(7, 3)$ code with $d_{min} = 4$. Therefore, it can correct any single error

  and detect any combination of double errors.

Encoding of Cyclic Codes

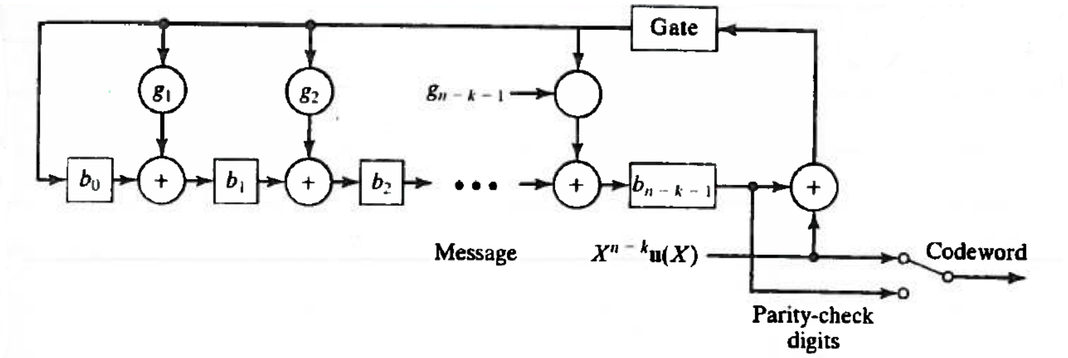
- We saw that if we multiply the information polynomial by X^{n-k} and divide by $g(X)$, we get:

$$X^{n-1}u(X) = a(X)g(X) + b(X)$$

and

$$a(X)g(X) = b(X) + X^{n-1}u(X)$$

is a codeword in systematic form. The following circuit encodes $u(X)$ based on the above discussion.



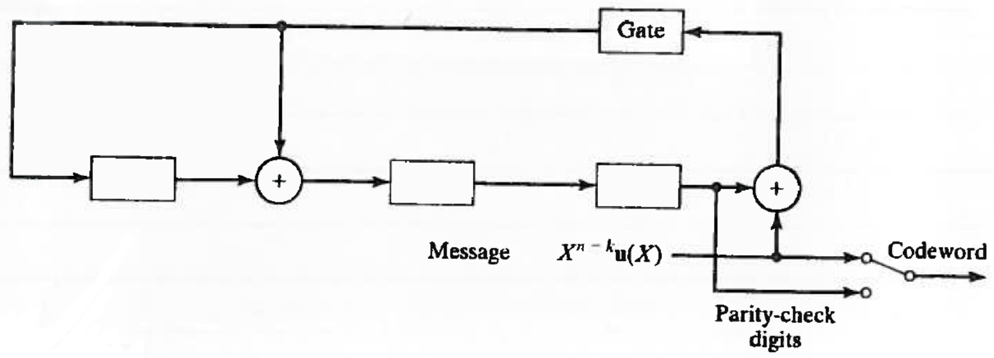
Encoding circuit for an (n, k) cyclic code with generator polynomial $g(X) = 1 + g_1X^2 + \cdots + g_{n-k-1}X^{n-k-1} + X^{n-k}$.

Encoding of Cyclic Codes

- The coding procedure is as follows:

1) Close the gate and enter information bits in and also send them over channel. This does multiplication by X^{n-k} as well as parity bit generation.
2) Open the gate (break the feedback).
3) Output the $n-k$ parity bits.

Example: $(7, 4)$ code with $g(X) = 1 + X + X^3$.



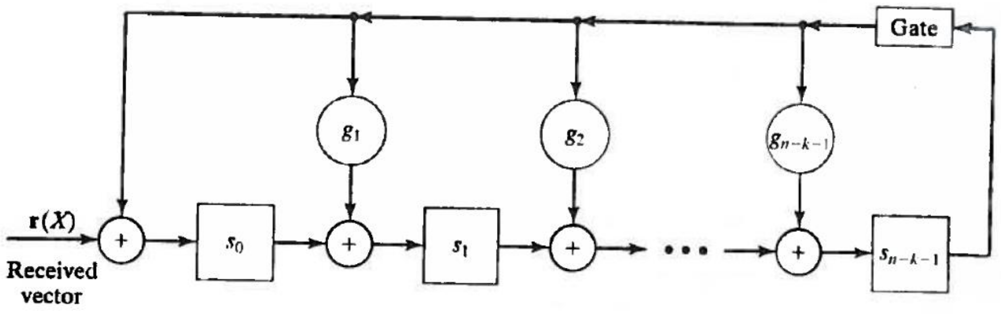
Encoder for the (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

Syndrome

- Assume $r(X) = r_0 + r_1X + r_2X^2 + \cdots + r_{n-1}X^{n-1}$ is the polynomial representing the received bits. Divide $r(X)$ by $g(X)$ to get:

$$r(X) = a(X)g(X) + s(X).$$

- $s(X)$ is a polynomial of degree $n-k-1$ or less. The $n-k$ coefficients of $s(X)$ are the syndromes.
- **Theorem 8:** let $s(X)$ be the syndrome of $r(X) = r_0 + r_1X + \cdots + r_{n-1}X^{n-1}$.
  Then, $s^{(i)}(X)$ resulting from dividing $X^i s(X)$ by $g(X)$ is the syndrome of $r^{(i)}(X)$.



An $(n-k)$-stage syndrome circuit with input from the left end.

Syndrome

- Example of $(7, 4)$ code:

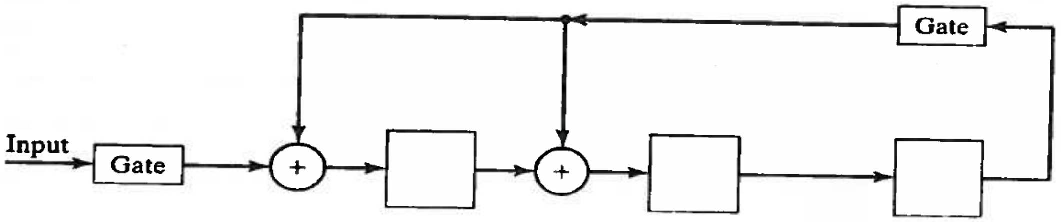
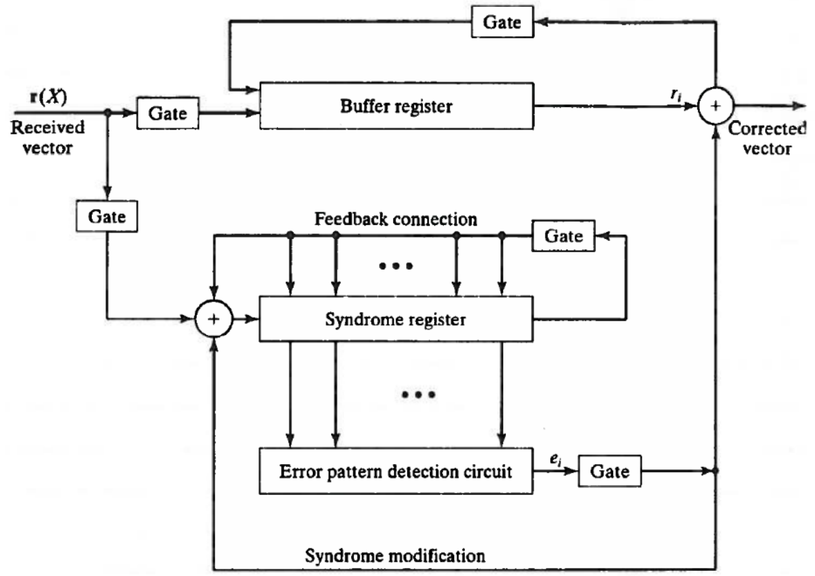


FIGURE 5.6: Syndrome circuit for the (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

TABLE 5.3: Contents of the syndrome register shown in Figure 5.6 with $\mathbf{r} = (0\,0\,1\,0\,1\,1\,0)$ as input.

Shift	Input	Register contents
		000 (initial state)
1	0	000
2	1	100
3	1	110
4	0	011
5	1	011
6	0	111
7	0	101 (syndrome $\mathbf{s}$)
8	—	100 (syndrome $\mathbf{s}^{(1)}$)
9	—	010 (syndrome $\mathbf{s}^{(2)}$)

Decoding of Cyclic Codes



General cyclic code decoder with received polynomial $\mathbf{r}(X)$ shifted into the syndrome register from the left end.

Decoding of Cyclic Codes

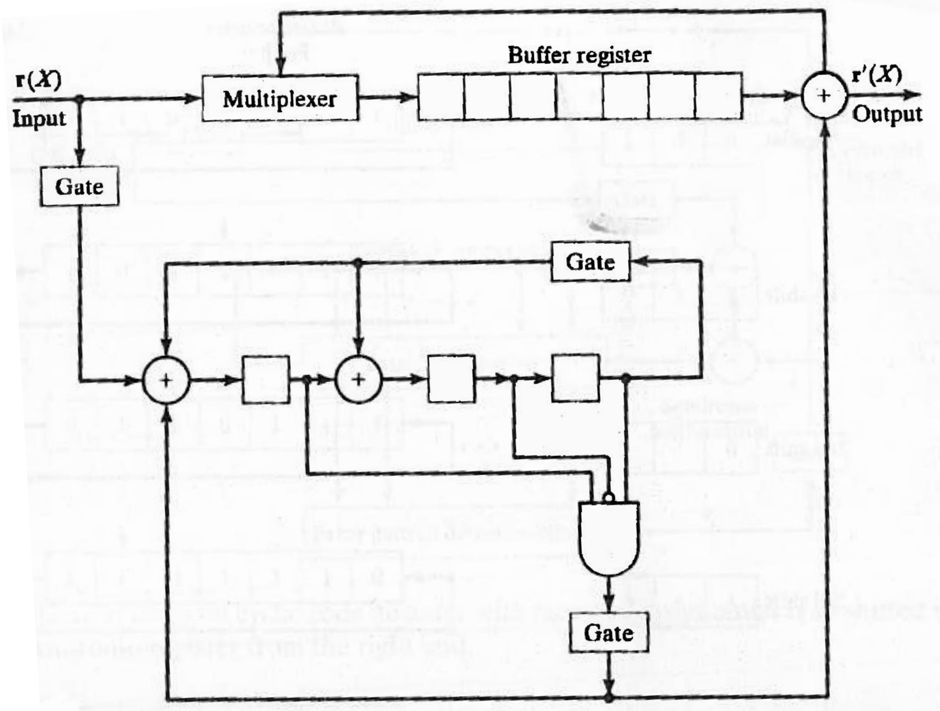
- Example: (7, 4) Hamming Code:

Error patterns and their syndromes with the received polynomial $\mathbf{r}(X)$ shifted into the syndrome register from the left end.

Error pattern $\mathbf{e}(X)$	Syndrome $\mathbf{s}(X)$	Syndrome vector (s_0, s_1, s_2)
$\mathbf{e}_6(X) = X^6$	$\mathbf{s}(X) = 1 + X^2$	(101)
$\mathbf{e}_5(X) = X^5$	$\mathbf{s}(X) = 1 + X + X^2$	(111)
$\mathbf{e}_4(X) = X^4$	$\mathbf{s}(X) = X + X^2$	(011)
$\mathbf{e}_3(X) = X^3$	$\mathbf{s}(X) = 1 + X$	(110)
$\mathbf{e}_2(X) = X^2$	$\mathbf{s}(X) = X^2$	(001)
$\mathbf{e}_1(X) = X^1$	$\mathbf{s}(X) = X$	(010)
$\mathbf{e}_0(X) = X^0$	$\mathbf{s}(X) = 1$	(100)

Decoding of Cyclic Codes

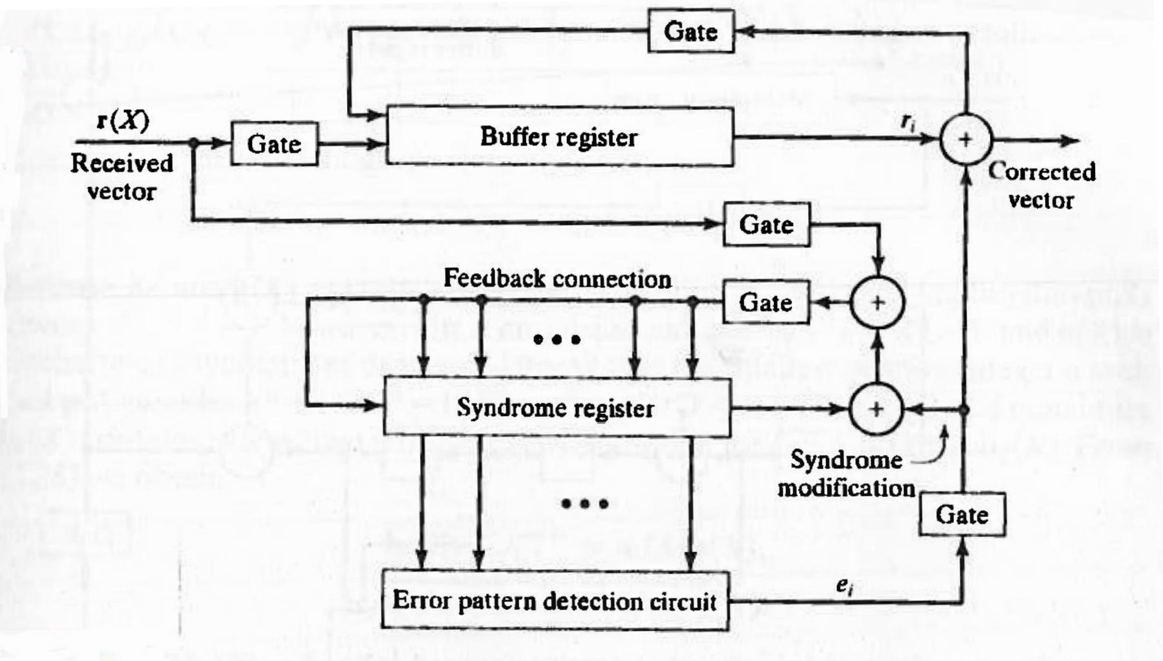
- Example: (7, 4) Hamming Code:



Decoding circuit for the (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

Decoding of Cyclic Codes

- **General Cyclic Code Decoder:**



General cyclic code decoder with received polynomial r(X) shifted into the syndrome register from the right end.

Decoding of Cyclic Codes

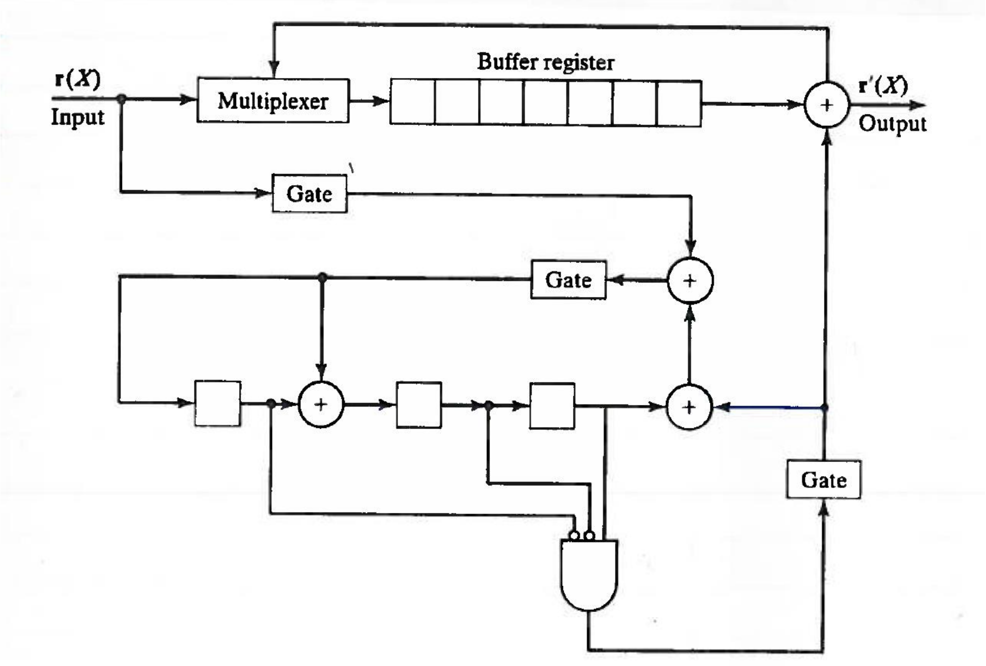
- Syndrome decoding of $(7, 4)$ code using syndrome decoder fed from right:

Error patterns and their syndromes with the received polynomial $\mathbf{r}(X)$ shifted into the syndrome register from the right end.

Error pattern $\mathbf{e}(X)$	Syndrome $\mathbf{s}^{(3)}(X)$	Syndrome vector (s_0, s_1, s_2)
$\mathbf{e}(X) = X^6$	$\mathbf{s}^{(3)}(X) = X^2$	(0 0 1)
$\mathbf{e}(X) = X^5$	$\mathbf{s}^{(3)}(X) = X$	(0 1 0)
$\mathbf{e}(X) = X^4$	$\mathbf{s}^{(3)}(X) = 1$	(1 0 0)
$\mathbf{e}(X) = X^3$	$\mathbf{s}^{(3)}(X) = 1 + X^2$	(1 0 1)
$\mathbf{e}(X) = X^2$	$\mathbf{s}^{(3)}(X) = 1 + X + X^2$	(1 1 1)
$\mathbf{e}(X) = X$	$\mathbf{s}^{(3)}(X) = X + X^2$	(0 1 1)
$\mathbf{e}(X) = X^0$	$\mathbf{s}^{(3)}(X) = 1 + X$	(1 1 0)

Decoding of Cyclic Codes

- Syndrome decoding of $(7, 4)$ code using syndrome decoder fed from right:



Decoding circuit for the (7, 4) cyclic code generated by $\mathbf{g}(X) = 1 + X + X^3$.

Cyclic Hamming Codes

- A Hamming code of length $n = 2^m - 1$ with $m \geq 3$ is generated by a primitive polynomial of degree m. let's see how we can put the Hamming code we discussed earlier in cyclic form:

Divide X^{m+i} by $p(X)$ to get $X^{m+i} = a_i(X)p(X) + b_i(X)$.

Since $p(X)$ is primitive, X is not a factor of $p(X)$ so $p(X)$ does not divide $X^{m+i} \Rightarrow b_i(X) \neq 0$.

- $b_i(X)$ has at least two terms. If it had one term:

$$X^{m+i} = a_i(X)p(X) + X^j$$
$$\Rightarrow X^j\left(X^{m+i-j} + 1\right) = a_i(X)p(X)$$

$$\Rightarrow p(X) \text{ divides } X^{m+i-j} + 1 \text{ but } m + i - j < 2^m - 1$$

$$\Rightarrow \text{contradiction.}$$

- If $i \neq j$, then $b_i(X) \neq b_j(X)$. Let

$$X^{m+i} = b_i(X) + a_i(X)p(X)$$
$$X^{m+j} = b_j(X) + a_j(X)p(X).$$

Cyclic Hamming Codes

- If $b_i(X) = b_j(X)$, then

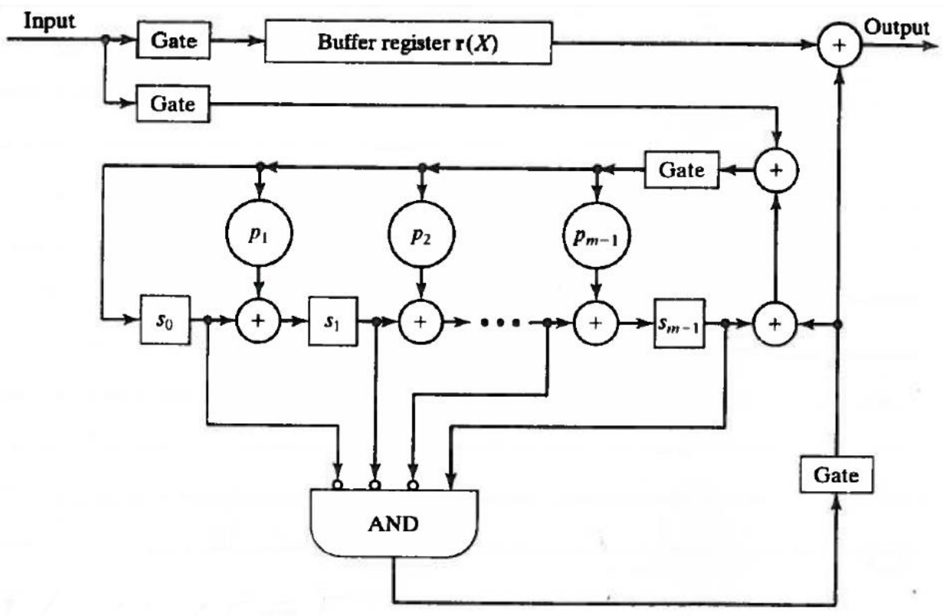
$$X^{m+i}(X^{j-i} + 1) = [a_i(X) + a_j(X)]p(X),$$

i.e., $p(X)$ divides $X^{j-i} + 1 \Rightarrow$ contradiction.

- Let $H = [I_m : Q]$ be the parity check matrix of this code. I_m is an $m \times m$ identity matrix with Q an $m \times (2^m - m - 1)$ matrix with $\underline{b}_i = (b_{i0}, b_{i1}, \cdots, b_{i,m-1})$ as its columns. Since no two columns of Q are the same and each has at least two 1's, then H is indeed a parity-check matrix of a Hamming code.

Syndrome Decoding of Hamming Codes

- Assume that error is in location with highest order, i.e.,
$$e(X) = X^{2^m-2}.$$
- Then, feeding $r(X)$ from right to syndrome calculator is equivalent to dividing $X^m \cdot X^{2^m-2}$ by the generator polynomial $p(X)$. Since $p(X)$ divides $X^{2^m-1} + 1$ then
$$s(X) = X^{m-1} \text{ or } \underline{s} = (0, 0, \cdots, 0, 1).$$



Decoder for a cyclic Hamming code.