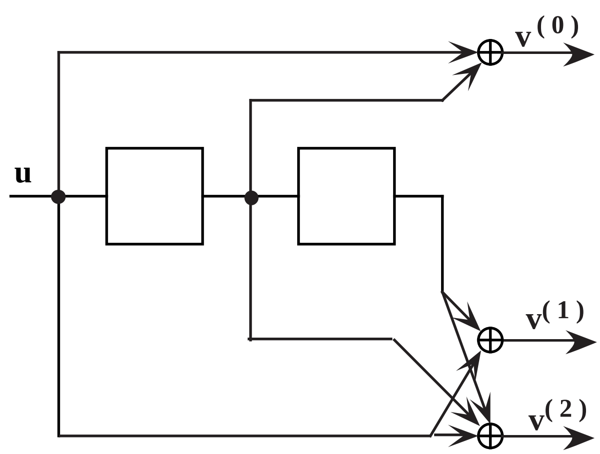
Chapter 11

Convolutional Codes

11.1 (a) The encoder diagram is shown below.



(b) The generator matrix is given by

$$\mathbf{G} = \begin{bmatrix} 111 & 101 & 011 & & & \\ & 111 & 101 & 011 & & \\ & & 111 & 101 & 011 & \\ & & & \ddots & & \ddots \end{bmatrix}.$$

(c) The codeword corresponding to $\mathbf{u} = (11101)$ is given by

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{G} = (111, 010, 001, 110, 100, 101, 011).$$

11.2 (a) The generator sequences of the convolutional encoder in Figure 11.3 on page 460 are given in (11.21).

(b) The generator matrix is given by

$$\mathbf{G} = \begin{bmatrix} 1111 & 0000 & 0000 & & \\ 0101 & 0110 & 0000 & & \\ 0011 & 0100 & 0011 & & \\ & 1111 & 0000 & 0000 & \\ & 0101 & 0110 & 0000 & \\ & 0011 & 0100 & 0011 & \\ & & \ddots & & \ddots \end{bmatrix}.$$

(c) The codeword corresponding to $\mathbf{u} = (110, 011, 101)$ is given by

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{G} = (1010, 0000, 1110, 0111, 0011).$$

11.3 (a) The generator matrix is given by

$$\mathbf{G}(D) = \begin{bmatrix} 1+D & 1+D^2 & 1+D+D^2 \end{bmatrix}.$$

(b) The output sequences corresponding to $\mathbf{u}(D) = 1 + D^2 + D^3 + D^4$ are

$$\begin{aligned} \mathbf{V}(D) &= \left[\mathbf{v}^{(0)}(D), \mathbf{v}^{(1)}(D), \mathbf{v}^{(2)}(D)\right] \\ &= \left[1 + D + D^2 + D^5,\ 1 + D^3 + D^5 + D^6,\ 1 + D + D^4 + D^6\right], \end{aligned}$$

and the corresponding codeword is

$$\begin{aligned} \mathbf{v}(D) &= \mathbf{v}^{(0)}(D^3) + D\mathbf{v}^{(1)}(D^3) + D^2\mathbf{v}^{(2)}(D^3) \\ &= 1 + D + D^2 + D^3 + D^5 + D^6 + D^{10} + D^{14} + D^{15} + D^{16} + D^{19} + D^{20}. \end{aligned}$$

11.4 (a) The generator matrix is given by

$$\mathbf{G}(D) = \begin{bmatrix} 1+D & D & 1+D \\ D & 1 & 1 \end{bmatrix}$$

and the composite generator polynomials are

$$\begin{aligned} \mathbf{g}_1(D) &= \mathbf{g}_1^{(0)}(D^3) + D\mathbf{g}_1^{(1)}(D^3) + D^2\mathbf{g}_1^{(2)}(D^3) \\ &= 1 + D^2 + D^3 + D^4 + D^5 \end{aligned}$$

and

$$\begin{aligned} \mathbf{g}_2(D) &= \mathbf{g}_2^{(0)}(D^3) + D\mathbf{g}_2^{(1)}(D^3) + D^2\mathbf{g}_2^{(2)}(D^3) \\ &= D + D^2 + D^3. \end{aligned}$$

(b) The codeword corresponding to the set of input sequences $\mathbf{U}(D) = \left[1 + D + D^3,\ 1 + D^2 + D^3\right]$ is

$$\begin{aligned} \mathbf{v}(D) &= \mathbf{u}^{(1)}(D^3)\mathbf{g}_1(D) + \mathbf{u}^{(2)}(D^3)\mathbf{g}_2(D) \\ &= 1 + D + D^3 + D^4 + D^6 + D^{10} + D^{13} + D^{14}. \end{aligned}$$

11.5 (a) The generator matrix is given by

$$\mathbf{G} = \begin{bmatrix} 111 & 001 & 010 & 010 & 001 & 011 & & & \\ & 111 & 001 & 010 & 010 & 001 & 011 & & \\ & & 111 & 001 & 010 & 010 & 001 & 011 & \\ & & & \ddots & & & & & \ddots \end{bmatrix}.$$

(b) The parity sequences corresponding to $\mathbf{u} = (1101)$ are given by

$$\begin{aligned} \mathbf{v}^{(1)}(D) &= \mathbf{u}(D) \cdot \mathbf{g}^{(1)}(D) \\ &= (1 + D + D^3)(1 + D^2 + D^3 + D^5) \\ &= 1 + D + D^2 + D^3 + D^4 + D^8, \end{aligned}$$

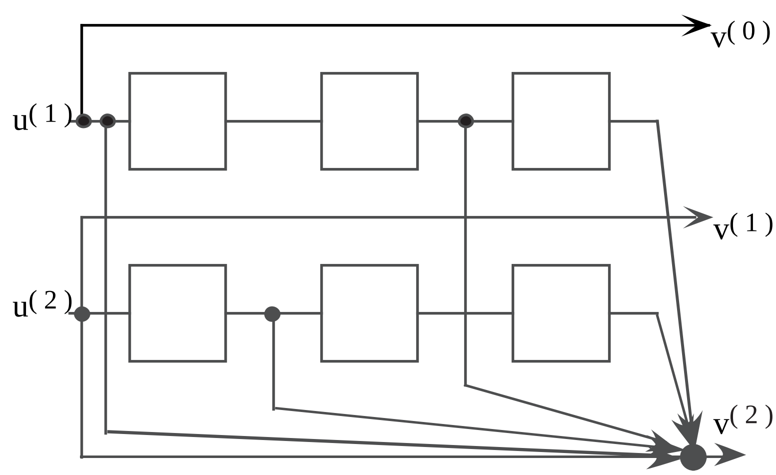
and

$$\begin{aligned} \mathbf{v}^{(2)}(D) &= \mathbf{u}(D) \cdot \mathbf{g}^{(2)}(D) \\ &= (1 + D + D^3)(1 + D + D^4 + D^5) \\ &= 1 + D^2 + D^3 + D^6 + D^7 + D^8. \end{aligned}$$

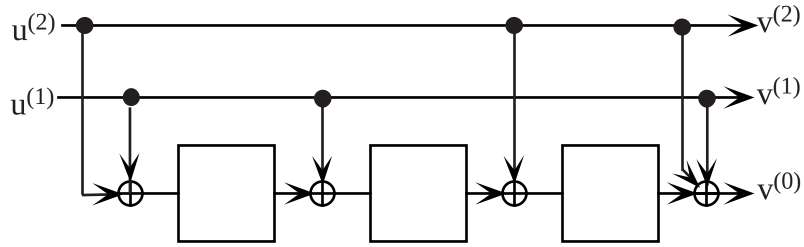
Hence,

$$\begin{aligned} \mathbf{v}^{(1)} &= (111110001) \\ \mathbf{v}^{(2)} &= (101100111). \end{aligned}$$

11.6 (a) The controller canonical form encoder realization, requiring 6 delay elements, is shown below.



(b) The observer canonical form encoder realization, requiring only 3 delay elements, is shown below.



11.14 (a) The GCD of the generator polynomials is 1.

(b) Since the GCD is 1, the inverse transfer function matrix $\mathbf{G}^{-1}(D)$ must satisfy

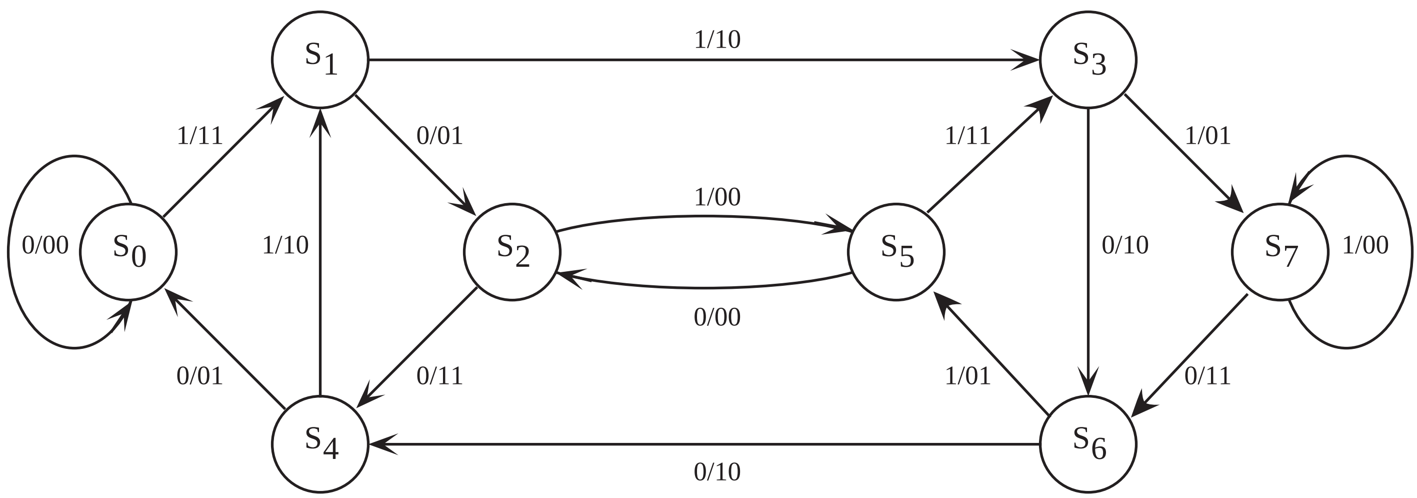
$$\mathbf{G}(D)\mathbf{G}^{-1}(D) = \left[1 + D^2 \quad 1 + D + D^2\right] \mathbf{G}^{-1}(D) = \mathbf{I}.$$

By inspection,

$$\mathbf{G}^{-1}(D) = \begin{bmatrix} 1 + D \\ D \end{bmatrix}.$$

11.15 (a) The GCD of the generator polynomials is $1 + D^2$ and a feedforward inverse does not exist.

(b) The encoder state diagram is shown below.



(c) The cycles $S_2 S_5 S_2$ and $S_7 S_7$ both have zero output weight.

(d) The infinite-weight information sequence

$$\mathbf{u}(D) = \frac{1}{1 + D^2} = 1 + D^2 + D^4 + D^6 + D^8 + \cdots$$

results in the output sequences

$$\begin{aligned} \mathbf{v}^{(0)}(D) &= \mathbf{u}(D)\left(1+D^2\right) = 1 \\ \mathbf{v}^{(1)}(D) &= \mathbf{u}(D)\left(1+D+D^2+D^3\right) = 1+D, \end{aligned}$$

and hence a codeword of finite weight.

(e) This is a catastrophic encoder realization.

11.16 For a systematic (n,k,ν) encoder, the generator matrix $\mathbf{G}(D)$is a $k \times n$ matrix of the form

$$\mathbf{G}(D) = [\mathbf{I}_k|\mathbf{P}(D)] = \begin{bmatrix} 1 & 0 & \cdots & 0 & \mathbf{g}_1^{(k)}(D) & \cdots & \mathbf{g}_1^{(n-1)}(D) \\ 0 & 1 & \cdots & 0 & \mathbf{g}_2^{(k)}(D) & \cdots & \mathbf{g}_2^{(n-1)}(D) \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \mathbf{g}_k^{(k)}(D) & \cdots & \mathbf{g}_k^{(n-1)}(D) \end{bmatrix}.$$

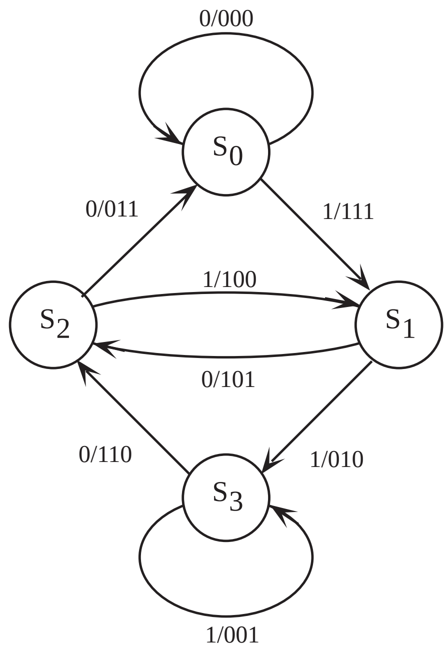
The transfer function matrix of a feedforward inverse $\mathbf{G}^{-1}(D)$ with delay $l=0$ must be such that

$$\mathbf{G}(D)\mathbf{G}^{-1}(D) = \mathbf{I}_k.$$

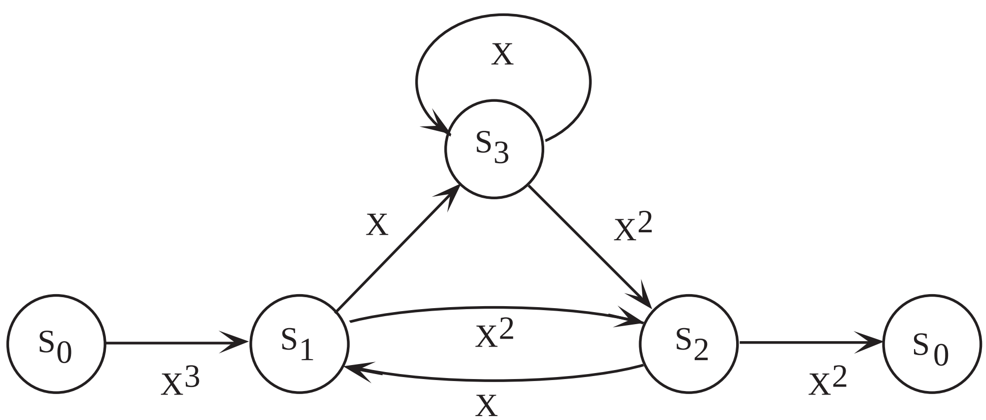
A matrix satisfying this condition is given by

$$\mathbf{G}^{-1}(D) = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0}_{(n-k)\times k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

11.19 (a) The encoder state diagram is shown below.



(b) The modified state diagram is shown below.



(c) The WEF function is given by

$$A(X) = \frac{\sum_i F_i \Delta_i}{\Delta}.$$

There are 3 cycles in the graph:

Cycle 1:	$S_1S_2S_1$	$C_1 = X^3$
Cycle 2:	$S_1S_3S_2S_1$	$C_2 = X^4$
Cycle 3:	S_3S_3	$C_3 = X$.

There is one pair of nontouching cycles:

$$\text{Cycle pair 1:} \quad \text{(Cycle 1, Cycle 3)} \quad C_1C_3 = X^4.$$

There are no more sets of nontouching cycles. Therefore,

$$\begin{aligned}\Delta &= 1 - \sum_i C_i + \sum_{i',j'} C_{i'}C_{j'} \\ &= 1 - (X + X^3 + X^4) + X^4 \\ &= 1 - X - X^3.\end{aligned}$$

There are 2 forward paths:

Forward path 1:	$S_0S_1S_2S_0$	$F_1 = X^7$
Forward path 2:	$S_0S_1S_3S_2S_0$	$F_2 = X^8$.

Only cycle 3 does not touch forward path 1, and hence

$$\Delta_1 = 1 - X.$$

Forward path 2 touches all the cycles, and hence

$$\Delta_2 = 1.$$

Finally, the WEF is given by

$$A(X) = \frac{X^7(1-X) + X^8}{1 - X - X^3} = \frac{X^7}{1 - X - X^3}.$$

Carrying out the division,

$$A(X) = X^7 + X^8 + X^9 + 2X^{10} + \cdots,$$

indicating that there is one codeword of weight 7, one codeword of weight 8, one codeword of weight 9, 2 codewords of weight 10, and so on.

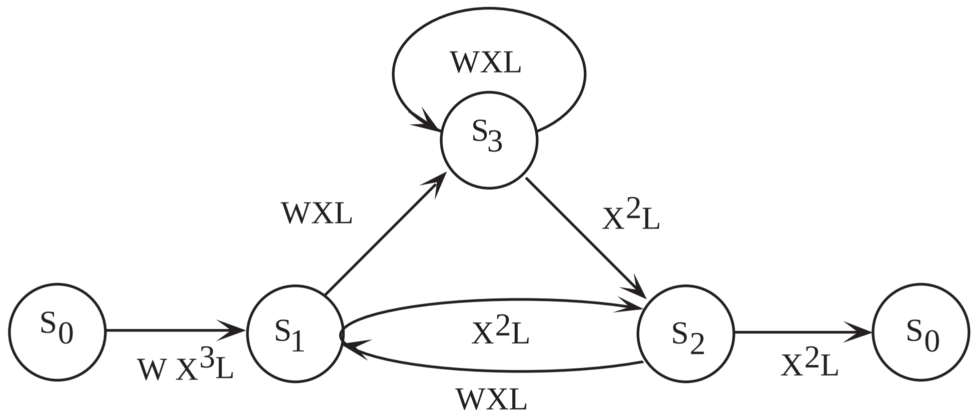
(d) The augmented state diagram is shown below.

(e) The IOWEF is given by

$$A(W, X, L) = \frac{\sum_i F_i \Delta_i}{\Delta}.$$

There are 3 cycles in the graph:

Cycle 1:	$S_1S_2S_1$	$C_1 = WX^3L^2$
Cycle 2:	$S_1S_3S_2S_1$	$C_2 = W^2X^4L^3$
Cycle 3:	S_3S_3	$C_3 = WXL$.



There is one pair of nontouching cycles:

$$\text{Cycle pair 1:} \quad (\text{Cycle 1, Cycle 3}) \quad C_1C_3 = W^2X^4L^3.$$

There are no more sets of nontouching cycles. Therefore,

$$\begin{aligned} \Delta &= 1 - \sum_i C_i + \sum_{i',j'} C_{i'}C_{j'} \\ &= 1 - WX^3L^2 + W^2X^4L^3 + WXL + W^2X^4L^3. \end{aligned}$$

There are 2 forward paths:

$$\begin{array}{lll} \text{Forward path 1:} & S_0S_1S_2S_0 & F_1 = WX^7L^3 \\ \text{Forward path 2:} & S_0S_1S_3S_2S_0 & F_2 = W^2X^8L^4. \end{array}$$

Only cycle 3 does not touch forward path 1, and hence

$$\Delta_1 = 1 - WXL.$$

Forward path 2 touches all the cycles, and hence

$$\Delta_2 = 1.$$

Finally, the IOWEF is given by

$$A(W,X,L) = \frac{WX^7L^3(1 - WXL) + W^2X^8L^4}{1 - (WX^3L^2 + W^2X^4L^3 + WXL) + X^4Y^2Z^3} = \frac{WX^7L^3}{1 - WXL - WX^3L^2}.$$

Carrying out the division,

$$A(W,X,L) = WX^7L^3 + W^2X^8L^4 + W^3X^9L^5 + \cdots,$$

indicating that there is one codeword of weight 7 with an information weight of 1 and length 3, one codeword of weight 8 with an information weight of 2 and length 4, and one codeword of weight 9 with an information weight of 3 and length 5.

11.20 Using state variable method described on pp. 505-506, the WEF is given by

$$A(X)=\frac{-X^4(-1-2X^4+X^3+9X^{20}-42X^{18}+78X^{16}+38X^{12}+3X^{17}+5X^{13}+9X^{11}-9X^{15}-2X^7-74X^{14}-14X^{10}-9X^9+X^6+6X^8)}{1-X+2X^4-X^3-X^2+3X^{24}+X^{21}-3X^{20}+32X^{18}-8X^{22}-X^{19}-45X^{16}-8X^{12}-5X^{17}-6X^{11}+9X^{15}+2X^7+27X^{14}+3X^{10}-2X^9-4X^6+X^8}.$$

Performing the division results in

$$A(X) = X^4 + X^5 + 2X^6 + 3X^7 + 6X^8 + 9X^9 + \cdots,$$

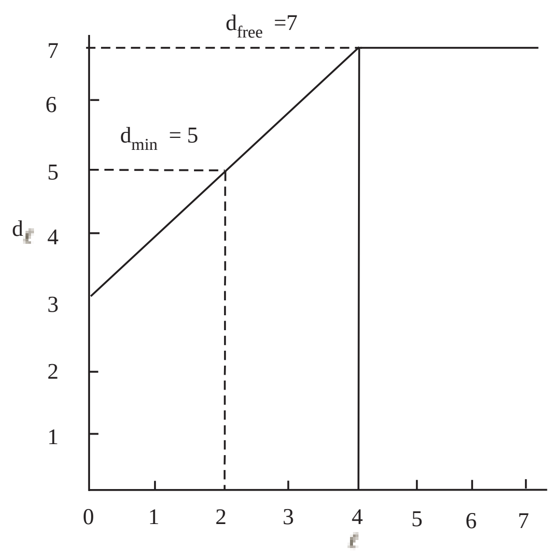
which indicates that there is one codeword of weight 4, one codeword of weight 5, two codewords of weight 6, and so on.

11.28 (a) From Problem 11.19(c), the WEF of the code is

$$A(X) = X^7 + X^8 + X^9 + 2X^{10} + \cdots,$$

and the free distance of the code is therefore $d_{free} = 7$, the lowest power of X in $A(X)$.

(b) The complete CDF is shown below.

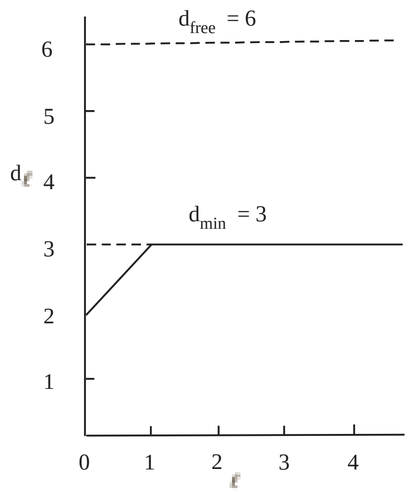


(c) The minimum distance is

$$d_{min} = d_l|_{l=m=2} = 5.$$

11.29 (a) By examining the encoder state diagram in Problem 11.15 and considering only paths that begin *and end* in state S_0 (see page 507), we find that the free distance of the code is $d_{free} = 6$. This corresponds to the path $S_0S_1S_2S_4S_0$ and the input sequence $\mathbf{u} = (1000)$.

(b) The complete CDF is shown below.



(c) The minimum distance is

$$d_{min} = d_l|_{l=m=3} = 3.$$

11.31 By definition, the free distance d_{free} is the minimum weight path that has diverged from and remerged with the all-zero state. Assume that $[\mathbf{v}]_j$ represents the shortest remerged path through the state diagram with weight free d_{free}. Letting $[d_l]_{re}$ be the minimum weight of all remerged paths of length l, it follows that $[d_l]_{re} = d_{free}$ for all $l \geq j$. Also, for a noncatastrophic encoder, any path that remains unmerged must accumulate weight. Letting $[d_l]_{un}$ be the minimum weight of all unmerged paths of length l, it follows that

$$\lim_{l\to\infty} [d_l]_{un} \to \infty.$$

Therefore

$$\lim_{l\to\infty} d_l = min\left\{\lim_{l\to\infty} [d_l]_{re}, \lim_{l\to\infty} [d_l]_{un}\right\} = d_{free}.$$

Q. E. D.