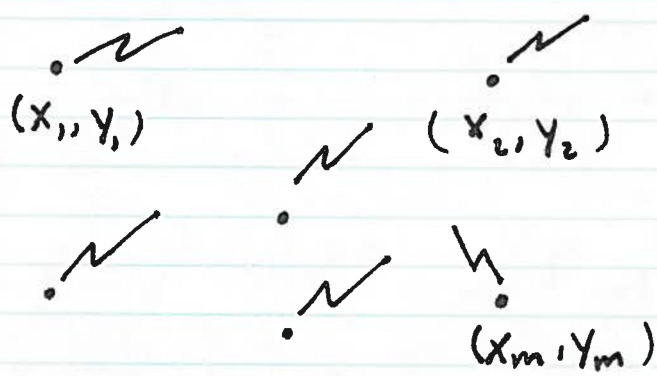
X Lecture 12, Nov. 18, 2003

Network information Theory:

general setting: m-nodes each transmitting X_i, $i \in \{1, \dots, m\}$ and each receiving Y_i, $i \in \{1, \dots, m\}$. The network (the set of channels) is characterized by $p(y^{(1)}, \dots, y^{(m)} | x^{(1)}, \dots, x^{(m)})$.

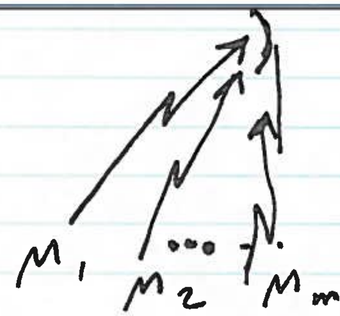


Special cases are:

- Broadcast Channel
- Multiple Access Channel
- Relay Channel.

Gaussian Multiple-Access channel

$$Y = \sum_{i=1}^{m} X_i + Z$$



If the SNR of each channel is $\frac{P}{N}$ then ~~each~~ no user cannot transmit at a rate higher than $\frac{1}{2}\log(1+\frac{P}{N})$. So,

example of mobile to base-station in cellular or Earth stations to Hub or satellite in Satcom.

$$R_i < \frac{1}{2}\log\left(1+\frac{P}{N}\right) \quad \forall i$$

similarly no two users can transmit at a rate higher than $\frac{1}{2}\log(1+\frac{2P}{N})$, i.e.,

$$R_i + R_j < \frac{1}{2}\log\left(1+\frac{2P}{N}\right) \quad \forall i, j$$

similarly,

$$R_i + R_j + R_k < \frac{1}{2}\log\left(1+\frac{3P}{N}\right)$$

$$\vdots$$

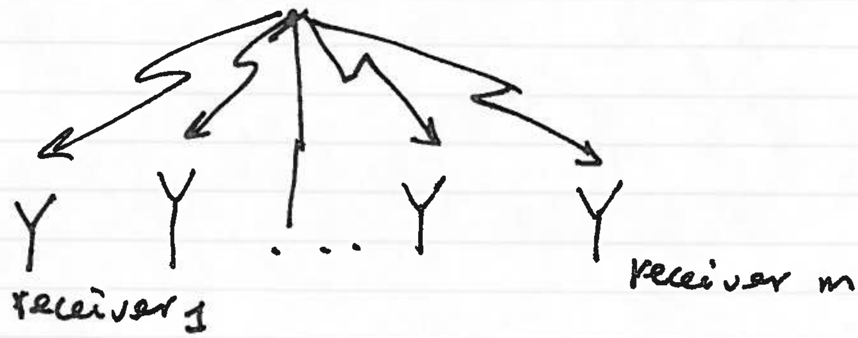
$$\sum_{i=1}^{m} R_i < \frac{1}{2}\log\left(1+\frac{mP}{N}\right)$$

Note: if the rates are the same, the last inequality dominates.

The Communication procedure:

- We have m Codebooks, one for each transmit e.g., a different Code for each user in a CDMA mobile. Each codebook has 2^{nR_i} Codewords, $i=1,\dots,m$.
- Each transmitter picks one of the Codewords say $X_i = X_i(k_i)$ $i \in \{1,\dots,m\}$, $k_i \in \{1,2,\dots,2^{nR_i}\}$
- The receiver (base) receives $\sum_{i=1}^{m} X_i$ plus a Gaussian Noise Component Z.
- The receiver minimizes the Euclidean distance between $\underline{Y}$ and various choices of Combinations of the m Codewords (one from each Codebook
- If $(R_1,\dots,R_m)$ belongs to the capacity region $R_i < \frac{1}{2}\log(1+\frac{P}{N}), \dots, \sum_{i=1}^{m} R_i < \frac{1}{2}\log(1+\frac{mP}{N})$ then the probability of error tends to zero.

Gaussian Broadcast Channel



- example of a TV station, a cellular base station sending control information common to all users. GPS sate
- example of a teacher or a speaker.

Simple example of a base sending with Power P and two receivers with noise powers N_1 and N_2 where $N_1 < N_2$.

So $\quad Y_1 = X + Z_1$

and

$$Y_2 = X + Z_2$$

where Z_1 and Z_2 are arbitrarily correlated Gaussian random variables with variances N_1 and N_2.

The transmitter wishes to send independent messages at rates R_1 and R_2 to receivers Y_1 and Y_2, e.g., a prof. in a cross-listed course.

The capacity region for this channel is

$$R_1 < \frac{1}{2}\log\left(1 + \frac{\alpha P}{N_1}\right)$$

and

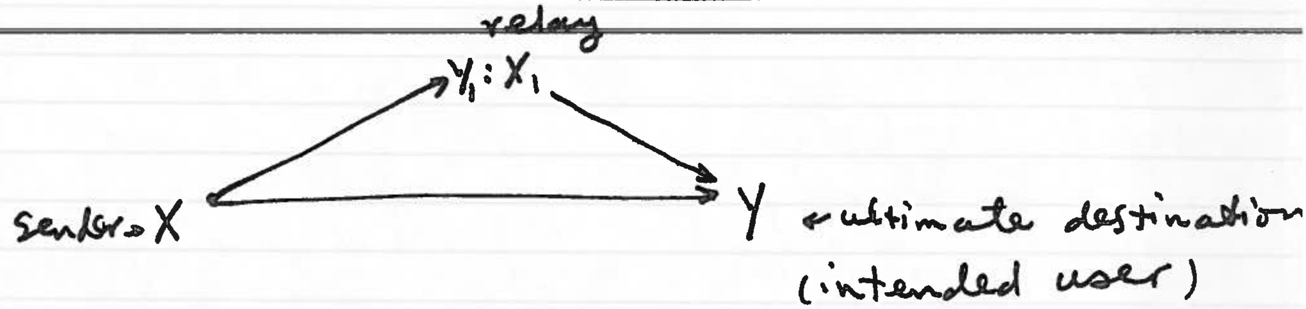
$$R_2 < \frac{1}{2}\log\left(1 + \frac{(1-\alpha)P}{\alpha P + N_2}\right)$$

$0 \leq \alpha \leq 1$ is ~~chosen~~ a design parameter.

The procedure is:

- The sender has two codes books: one with 2^{nR_1} codewords and power αP and the second with rate R_2 and power $(1-\alpha)P$
- at each transmission, the sender sends the sum of $X(i) \in \{1, \ldots, 2^{nR_1}\}$ and $X(j) \in \{1, \ldots, 2^{nR_2}\}$
- The weak receiver, searches $\{C_1, \ldots, C_{2^{nR_2}}\}$ codebook and picks the closest codeword. The αP acts as additional noise to him so his SNR is $\dfrac{(1-\alpha)P}{\alpha P + N_2}$.
- The strong receiver, first detects the message intended for the weak user, say $\hat{X}_2$ and then subtracts it from Y and work at a SNR of $\dfrac{\alpha P}{N_1}$.

Gaussian Relay Channel



$$Y_1 = X + Z_1$$

$$Y = X + Z_1 + X_1 + Z_2$$

where

$$X_{1i} = f_i(Y_{1,1}, Y_{1,2}, \ldots, Y_{1,i-1})$$

The sender has power P and relay has power P_1.

The capacity is:

$$\frac{1}{2}\log\left(1+\frac{\alpha P}{N}\right.$$

$$C = \max_{0 \le \alpha \le 1} \min\left\{ \frac{1}{2}\log\left(1+\frac{P+P_1+2\sqrt{(1-\alpha)PP_1}}{N_1+N_2}\right), \frac{1}{2}\log\left(\frac{\alpha P}{N_1}\right) \right\}$$

where $0 \le \alpha \le 1$ determines how much power is assigned to the two codebooks:

- One with rate R_1 and power αP such that $R_1 < \frac{1}{2}\log(1+\frac{\alpha P}{N_1})$,
- The other with rate R_0 and power $(1-\alpha)P$,

by the transmitter.

Transmission protocol:

- Sender sends a codewords from the first codebook (the one with rate R_1 and power αP) to both relay and the receiver.

- The relay correctly (with vanishing prob. of error) detects the codeword, since, $R_1 < \frac{1}{2}\log(1+\frac{\alpha}{N}$
- The receiver has an equivocation of
$R_1 - \frac{1}{2}\log(1+\frac{\alpha P}{N_1+N_2})$ and therefore a list of
$2^{n(R_1 - \frac{1}{2}\log(1+\frac{\alpha P}{N_1+N_2}))}$ codewords among which he has to decide.
- The remaining $(1-\alpha)P$ power from the sender and P_1 from relay is cooperatively used to transmit the partition of the 1st. codebook where the first codeword lies [The 1st codebook is partitioned into 2^{nR_0} sections (cells), this partition is known to all parties].
- In the next block the codeword in the second codebook corresponding to the cell in which

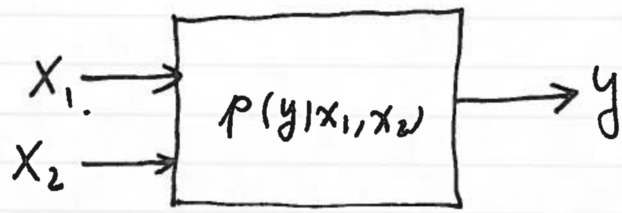
the first codeword was is sent with power $(\sqrt{(1-\alpha)P}+\sqrt{P_1})^2$ by sender and relay to the receiver.

- In the next block also, the sender sends ~~the~~ a new codeword (from the 1st codebook) added to the <u>cooperative</u> codeword.

- The receiver, first finds the index of the cooperative codeword and subtracts that ~~the~~ codeword from the received sequence (Y) and calculates a list of 2^{nR_0} codewords (of the 1st type) that may have been sent in the present (2nd) block and then intersects the list it had in the first block with the cell of partition it has learn in the second block and finds (with vanishing prob. of error) the codeword sent in the 1st. block. $\frac{P_1}{N_2} \geq \frac{P}{N_1}$

<u>Discussion of</u> rate $\frac{1}{2}\log(1+\frac{P_1}{N_1})$ achieved by letting $\alpha = 1$ and the no relay case of $\frac{1}{2}\log(1+\frac{P}{N_1+N_2})$, particularly when $N_2 \gg N_1$.

Definition: A discrete-memoryless multiple Access Consists of two input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, an output alphabet $\mathcal{Y}$, and a probability transition matrix $p(y|x_1, x_2)$.



Definition: A $((2^{nR_1}, 2^{nR_2}), n)$ Code Consists of two codebooks with $M_1 = 2^{nR_1}$ and $M_2 = 2^{nR_2}$ Codewords, respectively. Denote the first Codebook by

$$C_1 = \{ c_1^1, c_2^1, \dots, c_{2^{nR_1}}^1 \} \quad , \quad c_i^1 \in \mathcal{X}_1^n$$

and the second codebook

$$C_2 = \{ c_1^2, c_2^2, c_3^2, \dots, c_{2^{nR_2}}^2 \} \quad , \quad c_i^2 \in \mathcal{X}_2^n$$

Equivalently, the Code consists of two encoders (encoding functions:

$$f_1(\cdot) : \underbrace{\{1, \dots, 2^{nR_1}\}}_{W_1} \rightarrow \mathcal{X}_1^n$$

and

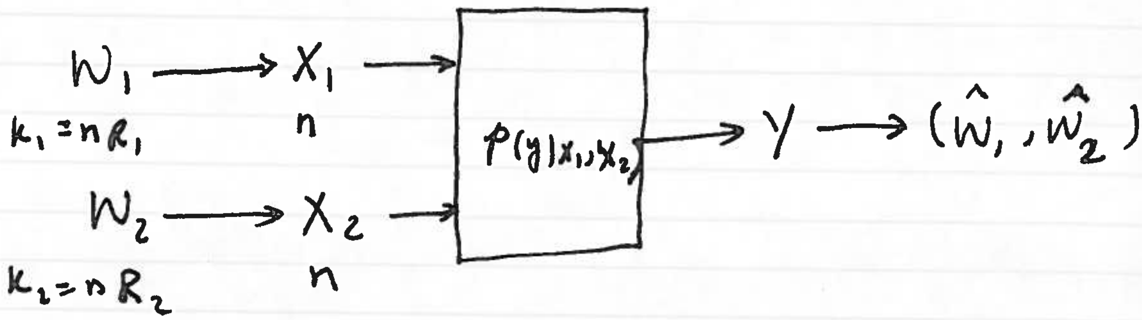
$$f_2(\cdot) : \underbrace{\{1, \dots, 2^{nR_2}\}}_{W_2} \rightarrow \mathcal{X}_2^n$$

The decoder is:

$$g : \mathcal{Y}^n \rightarrow \{1, \dots, 2^{nR_1}\} \times \{1, \dots, 2^{nR_2}\}$$

The probability of error is defined as:

$$P_e^{(n)} = \frac{1}{2^{n(R_1+R_2)}} \sum_{(w_1,w_2)\in W_1 \times W_2} \Pr\{ g(Y^n) \neq (w_1, w_2) \}$$



$$\frac{k_1}{n} = \frac{nR_1}{n} = R_1$$

$$\frac{k_2}{n} = \frac{nR_2}{n} = R_2$$

Definition: A rate pair (R_1, R_2) is said to be achievable if there is a sequence of $((2^{nR_1}, 2^{nR_2}), n)$ codes with $P_e^{(n)} \to 0$.

Definition: The capacity region of the multiple access channel is the set of all achievable (R_1, R_2) rate pairs.