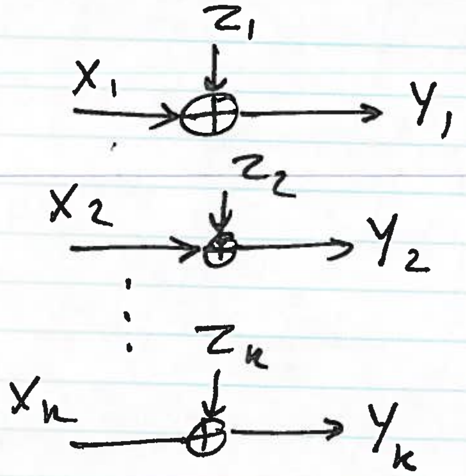
of 65000 bits gives $BER = 10^{-5}$ at $\frac{E_b}{N_0} = 0.7\,dB$ only 0.7 dB off the Shannon limit.

In the limit where $\frac{R}{W} \to 0$, we have

$$\lim_{\frac{R}{W}\to 0} \frac{E_b}{N_0} = \lim \frac{2^{R/W} \ln 2}{1} = \ln 2 = -1.6\,dB$$

Lecture 9, Oct. 28, 2003

Parallel Gaussian Channels:



$$Y_i = X_i + Z_i \qquad i = 1, 2, \dots k$$

$$Z_i \sim N(0, N_i)$$

w.r.t. constraint

$$E\left[\sum_{i=1}^{k} X_i^2\right] \leq P$$

$$C = \max_{f(x_1, \dots x_k) : \sum_i E X_i^2 \leq P} I(X_1, \dots X_k ; Y_1, \dots Y_k)$$

We have,

$$I(X_1,\cdots X_k;Y_1,\cdots Y_k)=h(Y_1,\cdots,Y_k)-h(Y_1,\cdots Y_k|X_1,\cdots)$$

$$=h(Y_1,\cdots,Y_k)-h(Z_1,\cdots,Z_k)=h(Y_1,\cdots,Y_k)-\sum_{i=1}^{k}H(Z_i)$$

$$\underset{(a)}{\leq}\sum_i[h(Y_i)-h(Z_i)]\underset{(b)}{\leq}\sum_i\frac{1}{2}\log(1+\frac{P_i}{N_i})$$

where $P_i=E[X_i^2]$ and $\sum_i P_i=P$.

Equality is achieved if (a) and (b) inequalities hold with equality. The inequality (a) hold iff Y_i's, consequently X_i's are ~~i.i.d.~~ independent and (b) holds iff they are Gaussian, i.e.,

$$(X_1,X_2,\cdots X_k)\sim N\left(\underline{0},\begin{bmatrix}P_1 & & 0\\ & P_2 & \\ 0 & & \ddots & P_k\end{bmatrix}\right)$$

Now, the problem is to maximize

$$\sum_i\frac{1}{2}\log(1+\frac{P_i}{N_i})$$

subject to $\sum_i P_i=P$. Let

$$J(P_1,\cdots,P_k)=\sum_i\frac{1}{2}\log(1+\frac{P_i}{N_i})+\mu(\sum_i P_i)$$

let $\frac{\partial J}{\partial P_i}=0$ $i=1,\cdots,k$ to get

$$\frac{1}{2}\frac{1}{P_i+N_i}+\mu=0$$

or $\quad P_i = \nu - N_i .$

Since P_i's have to be non-negative, it is not always possible to get a solution of this type. In such cases, we have

$$P_i = (\nu - N_i)^+ = \begin{cases} \nu - N_i & \nu \geq N_i \\ 0 & \nu < N_i \end{cases} = \max\{(\nu-$$

where ν is chosen such that

$$\sum_i (\nu - N_i)^+ = P$$

<u>Example</u>: Assume that two computers (or two switche are connected by two coaxial cable links. The first with 10 dB/km. loss terminated by a receiver with a noise ~~Figure~~ temperature of ~~10 dB~~ 3000 °K and the second with a loss of 11 dB/km. and a receiver with a noise ~~figure~~ temperature of ~~3 dB~~ 600 °K. The length of two cables is 10 km and the BW. is 1 MHz.

Then the loss in the first link is 100 dB, i.e.

$G_1 = 10^{-10}$ ~~and the noise temperature is $T_1 = (F-1)360$~~

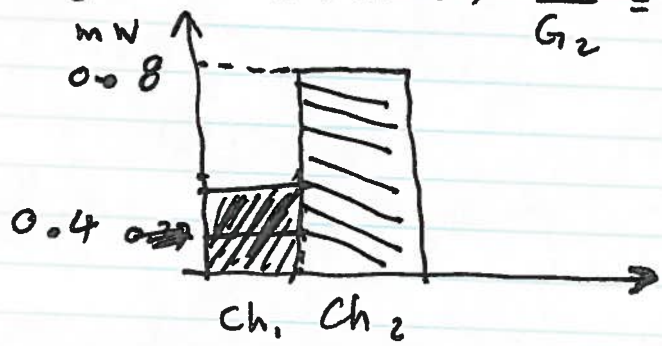
$T_1 = 3000\ °K \Rightarrow N_1 = KT_1B = 1.38\times10^{-23}\times 3000\times10^{6} \simeq 4\times$

$\frac{N_1}{G_1} = 4\times10^{-4} = 0.4$ mW

For the second cable Loss = 110 dB $\Rightarrow G_2 = 10^{-11}$

and $N_2 = 1.38\times10^{-23} \times T_2 \times 10^{6}$ where $T_2 = 600\,^{\circ}K$

$N_2 \cong 8\times10^{-15} \Rightarrow \dfrac{N_2}{G_2} = 8\times10^{-4}$



For this set of channels, we use the first channel only as long as ~~the~~ available power is less than 0.4 mW after that, we share the remaining power equally. So, if we got 1 mW of power, first, we ~~transmit~~ allocate 0.4 mW ~~in~~ to Ch1 and the remainder (0.6 mW) we divide between the two to get $P_1 = 0.7$ and $P_2 = 0.3$. Ther value of v is 1.1.

$P_1 = v - \hat{N}_1 = 1.1 - 0.4 = 0.7$ mW

$P_2 = v - \hat{N}_2 = 1.1 - 0.8 = 0.3$ mW

Channels with Colored Gaussian Noise (Correlated Gaussian Noise)

Let K_Z be the covariance matrix of the noise and K_X be the cov. matrix of the input.

The constraint $\frac{1}{n}\sum_i E[X_i^2] \leq P$

can be written as:

$$\frac{1}{n} tr(K_X) \leq P.$$

The mutual information can be written as

$$I(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = h(Y_1, \ldots, Y_n) - h(Z_1, \ldots, Z_n)$$

Since $h(Z_1, \ldots, Z_n)$ is independent of the input, we need to maximize $h(Y_1, \ldots, Y_n)$. The maximum value of $h(Y_1, \ldots, Y_n)$ is achieved when the output and, hence, the input is Gaussian.

Since Z and X are independent,

$$K_Y = K_X + K_Z.$$

So,

$$h(Y_1, \ldots, Y_n) = \frac{1}{2}\log\left((2\pi e)^K |K_X + K_Z|\right)$$

and

$$I(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = \frac{1}{2}\log\frac{|K_X + K_Z|}{|K_Z|}$$

or

$$I(X_1,\ldots X_n; Y_1,\ldots,Y_n) = \frac{1}{2}\log\det\left[I_n + K_X K_Z^{-1}\right]$$

Now, the problem is to choose K_X such that

$|K_X + K_Z|$ is maximized, subject to trace constraint $tr(K_X) \leq P$.

Let's decompose K_Z as:

$$K_Z = Q\Lambda Q^T$$

where Λ is a diagonal matrix $\Lambda = diag[\lambda_1,\ldots,\lambda$

and Q is a unitary matrix $QQ^T = I$

Then,

$$|K_X + K_Z| = |K_X + Q\Lambda Q^T|$$

$$= |Q||Q^T K_X Q + \Lambda||Q^T|$$

$$= |Q^T K_X Q + \Lambda|$$

$$= |A + \Lambda|$$

where $A = Q^T K_X Q$.

$$tr(A) = tr(Q^T K_X Q) = tr(QQ^T K_X) = tr(K_X)$$

So, now the problem is to maximize $|A+\Lambda|$ subject to the constraint $tr(A) \leq nP$.

111

According to Hadamard's inequality for any positive definite matrix K, we have

$$|K| \leq \prod_i K_{ii}$$

where K_{ii} are the diagonal elements of K.

So,

$$|A+\Lambda| \leq \prod_i (A_{ii}+\lambda_i)$$

with equality iff A is diagonal.

Since A is subject to the trace constraint

$$\frac{1}{n}\sum_i A_{ii} \leq P$$

and $A_{ii} \geq 0$, the maximum value of $\prod_i (A_{ii}+\lambda_i)$ is achieved when

$$A_{ii}+\lambda_i = \nu .$$

Given the constraints, there may be cases when this equation cannot be satisfied with positive A_{ii}. In such cases, we use the Kuhn-Tucker conditions that the optimum solution is:

$$A_{ii} = (\nu - \lambda_i)^+$$

where ν is chosen so that $\sum_i A_{ii} = nP$.

Then, the capacity is

$$C = \max_{f(\underline{X}):\frac{1}{n}tr(K_{\underline{X}})\le P} I(X_1,\dots,X_n; Y_1,\dots Y_n)$$

$$= \max_{\frac{1}{n}tr(K_X)\le P} \frac{1}{2}\log \frac{|K_X+K_Z|}{|K_Z|} = \max_{\frac{1}{n}tr(A)\le P} \frac{1}{2}\log\frac{|A+\Lambda|}{|\Lambda|}$$

$$= \frac{1}{2}\log \frac{\prod_{i=1}^{n}((\mu-\lambda_i)^+ + \lambda_i)}{\prod_{i=1}^{n}\lambda_i} = \frac{1}{2}\log\prod_{i=1}^{n}\left(1+\frac{(\mu-\lambda_i)^+}{\lambda_i}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\log\left(1+\frac{(\mu-\lambda_i)^+}{\lambda_i}\right)$$

where μ is chosen such that

$$\sum_{i=1}^{n}(\mu-\lambda_i)^+ = nP.$$

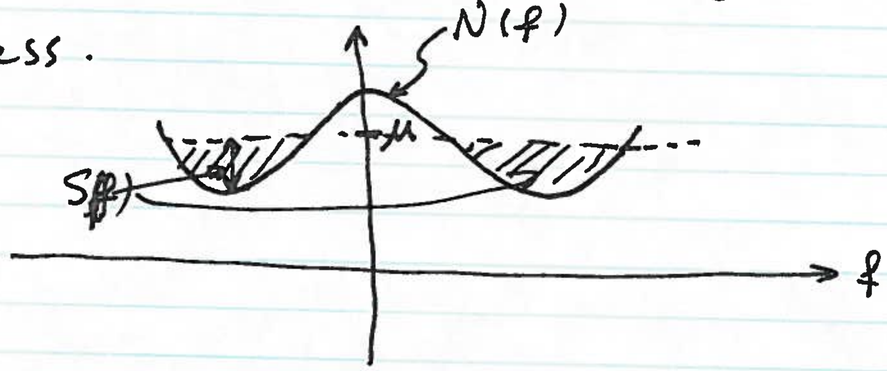
For a waveform channel (continuous-time chan when the noise is a random process with covarian matrix $K_Z^{(n)}$. If the process is stationary, the covariance matrix is Toeplitz and as $n\to\infty$, eigenvalues tend to a limit. The density of eigenvalues on the real line, tend to the power spectrum of the process. Then, we get

$$C = \int \frac{1}{2}\log\left(1+\frac{(\nu-N(f))^+}{N(f)}\right)df$$

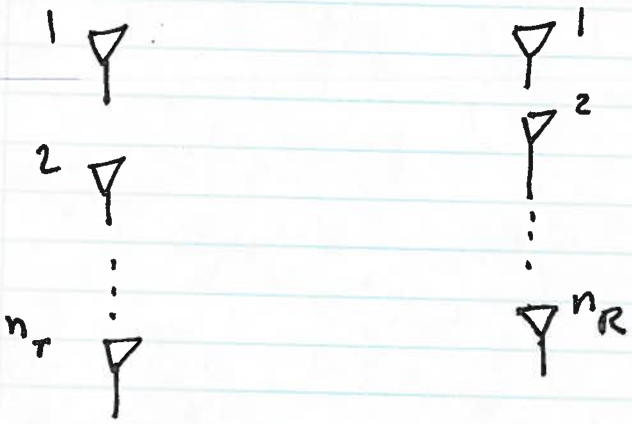
where ν is chosen such that

$$\int (\nu - N(f))^+ df = P$$

Here $N(f)$ is the spectral density of the noise process.



Capacity of MIMO Channels:



Consider a multiple input - multiple output channel with n_T transmit antennas and n_R receive antennas. Each receiver has a noise with power N and the total transmit power, irrespective of the number of transmit antennas: $\hat{P}$. The received signal is $y(t) = g(t) * s(t) + z(t)$

a matrix with n_T col and n_R rows

we assume that the channel is frequency flat, so, we can make narrow-band assumption, i.e., $g(0)$ is non-zero and $g(t)$ is a zero matrix for other values of t. So,

$$y(t) = g(0) \cdot S(t) + z(t)$$

in vector form:

$$\underline{y} = G\underline{s} + \underline{z} = \underline{x} + \underline{z} \qquad (1)$$

where $E[s^{\dagger}s] = \hat{P}$

when there is one transmit antenna, we transmit $\hat{P}$ over that antenna and the average output power of each receiver is P. The SNR is $\rho = \frac{P}{N}$.

we can re-write in terms of another matrix H related to $\underline{G}$ as:

$$\hat{P}|G|^2 = P|H|^2$$

or $$\hat{P}^{1/2} \cdot G = P^{1/2} \cdot H$$

Then

$$\underline{y} = \left(\frac{P}{\hat{P}}\right)^{1/2} H \cdot s + \underline{z} = \underline{x} + \underline{z}$$

$$C = \frac{1}{2} \log \frac{|K_x + K_z|}{|K_z|}$$

$$K_z = N I_{nR}$$

$$|K_x + K_z| = \left| E\left\{ \frac{P}{\hat{P}} H s s^{\dagger} H^{\dagger} + z z^{\dagger} \right\} \right|$$

$$= \left| \frac{P}{\hat{P}} H E[s s^{\dagger}] H^{\dagger} + N I_{nR} \right| = \left| \frac{P}{\hat{P}} H K_s H^{\dagger} + N \ldots \right.$$

~~equality holds if components of s are~~ $\cancel{\text{iid}}$ ~~normal~~

$$= \left| \frac{P}{n_T} H H^{\dagger} + N I_{nR} \right| \quad \text{assuming } K_s = \begin{bmatrix} \hat{P}/n_T & \\ 0 & \ddots \end{bmatrix}$$

$$C = \frac{1}{2} \log \det \left[I_{nR} + \frac{P}{n_T N} H H^{\dagger} \right]$$

$$C = \frac{1}{2} \log \det \left[I_{nR} + \frac{\rho}{n_T} H H^{\dagger} \right] \quad \text{bits/use}$$

or

$$C = \log \det \left[I_{nR} + \frac{\rho}{n_T} H H^{\dagger} \right] \quad \text{bits/sec./Hz.}$$

Take the case of $n \times n$ MIMO: $(n_T = n_R = n)$

Then
$$C = \log \det \left[I_n + \frac{\rho}{n} H H^{\dagger} \right]$$

let $H = I_n$, then

$$C = \log \det \left[I_n \left(1 + \frac{\rho}{n}\right) \right] = \log \left(1 + \frac{\rho}{n}\right)^n = n \log\left(1 + \frac{\rho}{n}\right)$$

as $n \to \infty$ $\quad C = n \log\left(1 + \frac{\rho}{n}\right) \to \frac{\rho}{\ln(2)}$

So Capacity increases linearly (not logarithmically) with SNR.

$$C = C(H) = \log \det \left[I_{n_R} + \frac{\rho}{n_T} H H^{+} \right]$$

is the capacity for a given realization of the channel (H). Usually CCDFs (Complement Cumulative Dist. Function) of the capacity is found. For example for an outage value of 0.01, one calculates 99% CCDF, i.e., the capacity that 99% of the times is exceeded.

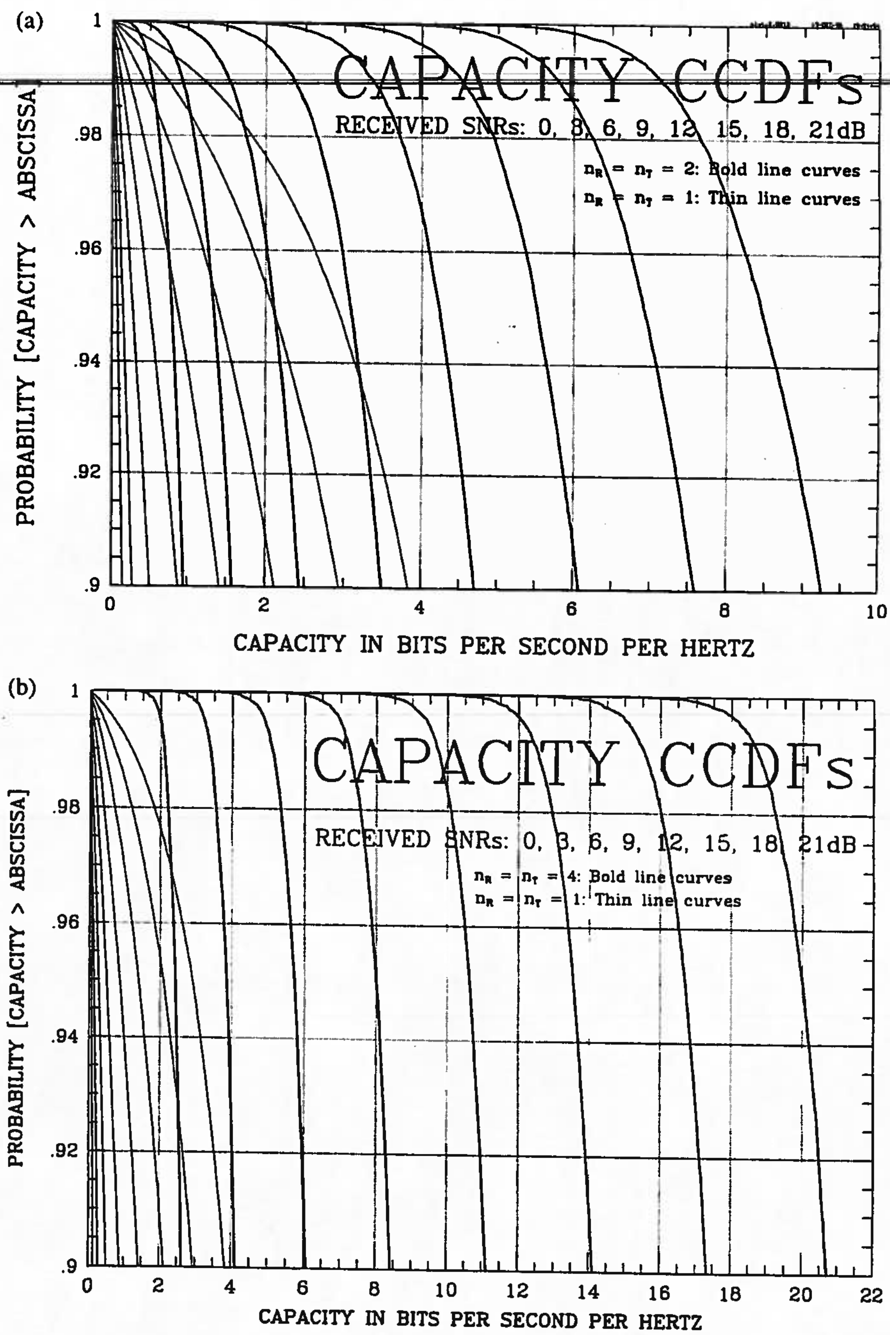


Figure 1. Capacity: Complementary Cumulative Distribution Functions. Assumes statistically independent Rayleigh faded paths. Average received SNR is a parameter ranging from 0 to 21 dB in steps of 3 dB. (a) Two antennas at both transmitter and receiver (bold line curves). Single antenna at both transmitter and receiver shown for reference (thin line curves). (b) Same as (a) except four antennas depicted by bold line curves.