Assignment # 1

P1 – The noise voltage in an electric circuit can be modeled as a Gaussian random variable with mean equal to zero and variance equal to 10^{-8}.

1. What is the probability that the value of the noise exceeds 10^{-4}? What is the probability that it exceeds 4×10^{-4}? What is the probability that the noise value is between -2×10^{-4} and 10^{-4}?
2. Given that the value of the noise is positive, what is the probability that it exceeds 10^{-4}?
3. This noise passes through a half-wave rectifier with characteristics

$$g(x) = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

   Find the PDF of the rectified noise by first finding its CDF. Why can we not use the general expression in Equation (4.1.10) here?
4. Find the expected value of the rectified noise in the previous part.
5. Now assume that the noise passes through a full-wave rectifier defined by $g(x) = |x|$. Find the density function of the rectified noise in this case. What is the expected value of the output noise in this case?

P2 – Let Y be a positive valued random variable; i.e., $f_Y(y) = 0$ for $y < 0$.

1. Let α be any positive constant. Show that $P(Y > \alpha) \leq \frac{E[Y]}{\alpha}$ (Markov inequality).
2. Let X be any random variable with variance σ^2 and define $Y = (X - E[X])^2$ and $\alpha = \epsilon^2$ for some ϵ. Obviously the conditions of the problem are satisfied for Y and α as chosen here. Derive the Chebychev inequality

$$P(|X - E[X]| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

P3 – Show that for a binomial random variable, the mean is given by np and the variance is given by $np(1-p)$.

P4 – Two random variables X and Y are distributed according to

$$f_{X,Y}(x, y) = \begin{cases} Ke^{-x-y}, & x \geq y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the value of the constant K.
2. Find the marginal density functions of X and Y.
3. Are X and Y independent?
4. Find $f_{X|Y}(x \mid y)$.
5. Find $E[X \mid Y = y]$.
6. Find $\text{COV}(X, Y)$ and $\rho_{X,Y}$.

P5 – Random variables X and Y are jointly Gaussian with

$$\mathbf{m} = [1 \quad 2]$$

$$\mathbf{C} = \begin{bmatrix} 4 & -4 \\ -4 & 9 \end{bmatrix}$$

1. Find the correlation coefficient between X and Y.
2. If $Z = 2X + Y$ and $W = X - 2Y$, find $\text{COV}(Z, W)$.
3. Find the PDF of Z.

P6 — Two random variables X and Y are distributed according to

$$f_{X,Y}(x, y) = \begin{cases} \frac{K}{\pi} e^{-\frac{x^2+y^2}{2}}, & xy \geq 0 \\ 0, & xy < 0 \end{cases}$$

1. Find K.
2. Show that X and Y are each Gaussian random variables.
3. Show that X and Y are not jointly Gaussian.
4. Are X and Y independent?
5. Are X and Y uncorrelated?
6. Find $f_{X|Y}(x|y)$. Is this a Gaussian distribution?

P7 — Which one of the following functions can be the autocorrelation function of a random process and why?

1. $f(\tau) = \sin(2\pi f_0 \tau)$.
2. $f(\tau) = \tau^2$.
3. $f(\tau) = \begin{cases} 1 - |\tau| & |\tau| \leq 1 \\ 1 + |\tau| & |\tau| > 1 \end{cases}$
4. $f(\tau)$ as shown in Figure P-4.40.

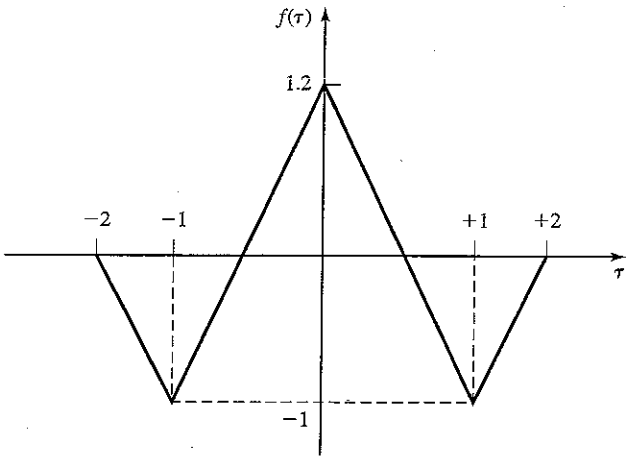

Figure P-4.40

P8 — The random process $X(t)$ is defined by

$$X(t) = X \cos 2\pi f_0 t + Y \sin 2\pi f_0 t$$

where X and Y are two zero-mean independent Gaussian random variables each with variance σ^2.

1. Find $m_X(t)$.
2. Find $R_X(t + \tau, t)$. Is $X(t)$ stationary? Is it cyclostationary?
3. Find the power-spectral density of $X(t)$.
4. Answer the above questions for the case where $\sigma_X^2 = \sigma_Y^2$