ELEC 6831

Digital Transmission Systems

Lecture 1

- Course outline

- Review of Probability and random processes.

Relative Frequency versus the axiomatic definition of Probability.

- Sample Space $S = \{s_1, s_2, \ldots, \ldots\}$: a set of events
- $P[A]$: A mapping from subsets of S to $[0, 1]$ such that ;

i) $0 \leq P[A] \leq 1 \quad \forall A \in S$

ii) $P[S] = 1$

iii) $P[A \cup B] = P[A] + P[B]$

if $A \cap B = \phi$

If $A \cap B \neq \phi$ then

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Conditional Probability

$$P[B|A] = \frac{P[A \cap B]}{P[A]}$$

or

$$P[A \cap B] = P[A] P[B|A]$$

similarly

$$P[A \cap B] = P[B] P[A|B]$$

So,

$$P[A] P[B|A] = P[B] P[A|B]$$

or

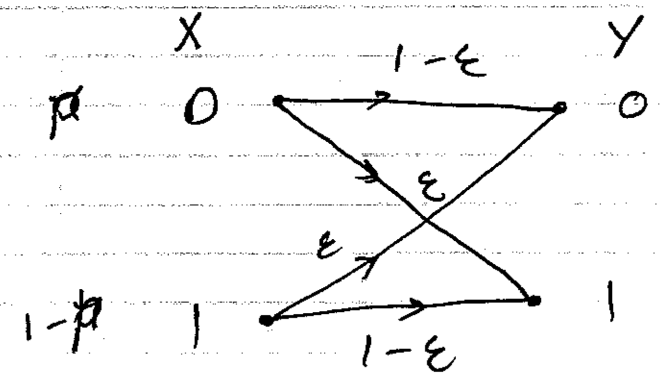
$$P[A|B] = \frac{P[A] P[B|A]}{P[B]}$$

when $P[A|B] = P[A]$, we have:

$$P[A \cap B] = P[A] P[B]$$

and say A and B are independent.

Binary Symmetric Channel (BSC):



$$P[Y=0] = p(1-\varepsilon) + (1-p)\varepsilon$$

$$P[Y=1] = p\varepsilon + (1-p)(1-\varepsilon)$$

So,

$$P[X=0 \mid Y=0] = \frac{P[X=0]\,P[Y=0 \mid X=0]}{P[Y=0]}$$

$$= \frac{p(1-\varepsilon)}{p(1-\varepsilon)+(1-p)\varepsilon}$$

$$P[X=1 \mid Y=1] = \frac{P[X=1]\,P[Y=1 \mid X=1]}{P[Y=1]}$$

$$= \frac{(1-p)(1-\varepsilon)}{p\varepsilon+(1-p)(1-\varepsilon)}$$

Random Variables

A random variable X or $X(s)$ is a function from S to $\mathbb{R}$, i.e., it maps each $s_i \in S$ to a real value $X(s_k) = x$.

Cumulative Distribution Function (CDF)

$$F_X(x) = P[X \leq x]$$

it is clear that

I) $$0 \leq F_X(x) \leq 1$$ since it is a probability

II) $$F_X(-\infty) = 0$$
$$F_X(\infty) = 1$$

III) $$P[x_1 \leq X \leq x_2] = F_X(x_2) - F_X(x_1)$$

Probability Density Function (pdf)

$$f_X(x) = \frac{d}{dx} F_X(x)$$

So,

$$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$$

therefore,

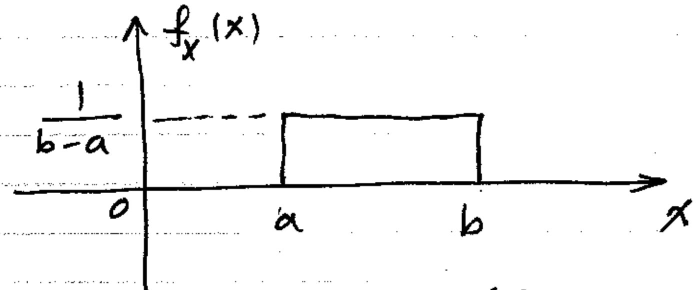
$$P[X \leqslant x_i] = \int_{-\infty}^{x_i} f_X(x)\,dx$$

and,

$$P[x_1 \leqslant X \leqslant x_2] = F_X(x_2) - F_X(x_1)$$

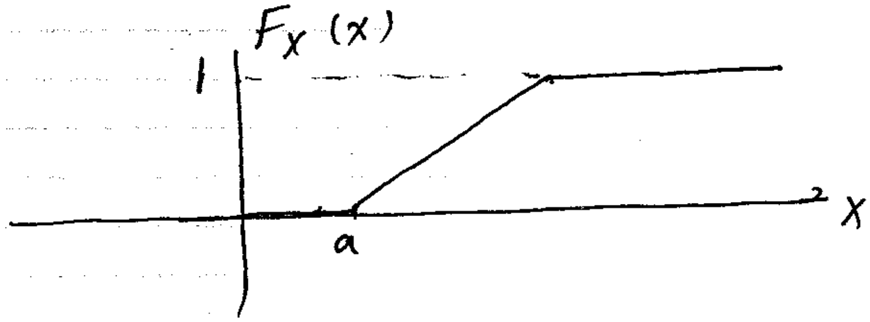
$$= \int_{x_1}^{x_2} f_X(x)\,dx$$

Uniform Distribution:



$$f_X(x) = \begin{cases} 0 & x \leqslant a \\ \dfrac{1}{b-a} & a < x \leqslant b \\ 0 & x > b \end{cases}$$

So,



$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x \le b \\ 1 & x > b \end{cases}$$

Note that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Multiple random variables:

$$F_{X,Y}(x,y) = P[X \le x,\ Y \le y]$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(\xi,\eta)\,d\xi\, d\eta = 1$$

and

$$F_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X,Y}(\xi,\eta)\,d\xi\, d\eta$$

and

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,\eta)\,d\eta$$

Conditional probability density function

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Mean: (average)

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Similarly if

$Y = g(X)$ is a function of X

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Example

$$Y = \cos(X)$$

where

$$f_X(x) = \begin{cases} 1/2\pi & -\pi < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\pi}^{\pi} \cos(x) \frac{1}{2\pi}\,dx = -\frac{1}{2\pi} \sin(x)\Big|_{-\pi}^{\pi} = 0$$

Moments

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

is called the n-th moment of the r.v. X.

For $n=1$, we get the mean.

For $n = 2$, we have $E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

Central moments

$$E[(X-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x)\,dx$$

for $n=2$, we have

$$E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

this is called the variance.

$$Var(X) = \sigma_X^2 = E[(X-\mu_X)^2]$$

σ_X (the square-root of the variance) is called the standard deviation.

For $n=1$ the central moment is equal to zero, i.e., subtracting the mean results in zero-mean (as expected!!).

Note that:

$$\begin{aligned}\sigma_X^2 = E[(X-\mu_X)^2] &= E[X^2] - E[2\mu_X X] \\ &\quad + E[\mu_X^2] \\ &= E[X^2] - 2\mu_X E[X] + \mu_X^2 \\ &= E[X^2] - \mu_X^2.\end{aligned}$$

Chebyshev's inequality

$$P[\,|X-\mu_X| \geq \varepsilon] \leq \frac{\sigma_X^2}{\varepsilon^2}$$

is an indication of the importance of mean and variance in limiting the region where we expect to observe the result of an experiment.

Covariance

$$Cov[X,Y] = E[(X-E(X))(Y-E(Y))]$$

$$= E[XY] - \mu_X \mu_Y$$

The correlation coefficient

$$\rho = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}$$

if $Cov[X,Y]=0$ we say X and Y are uncorrelated.

if $E[XY]=0$ we say X and Y are orthogona

Random Processes

A random Process - is an assignment to a function (a time function or signal) to each outcome of a sample space.

$$X(t, s) \qquad -T \leq t \leq T$$

Mean

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$$

if $\quad f_{X(t_1)}(x) = f_{X(t_2)}(x)$

then

$$\mu_X(t) = \mu_X \qquad \text{for all } t$$

and we say that $X(t)$ is first order stationary.

A Second order stationary random process has all $f_{X(t_1)X(t_2)}(x_1, x_2)$ that only depend on $t_2 - t_1$.

A random process is stationary (in the strict sense) if all it statistics are time independent (shift independent).

Weak sense (or wide sense) stationary (WSS)

if $\mu_X(t) = \mu_X \quad \forall t$

and

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \, f_{X(t_1)X(t_2)}(x_1, x_2)\, dx_1\, dx_2 = R_X(t_2 - t_1)$$

then $X(t)$ is called WSS.

Properties of the autocorrelation function

For a stationary process:

$$R_X(\tau) = E[X(t+\tau)X(t)] \quad \forall t$$

1)

$$R_X(0) = E[X^2(t)]$$

2)

$$R_X(\tau) = R_X(-\tau)$$

3)

$$|R_X(\tau)| \leq R_X(0)$$

Proof:

$$E[\{X(t+\tau) \pm X(t)\}^2] \geq 0$$

$$E[X^2(t+\tau)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t)] \geq 0$$

$$2R_X(0) \pm 2R_X(\tau) \geq 0$$

$$-R_X(0) \leq R_X(\tau) \leq R_X(0)$$

Example:

$$X(t) = A\cos(2\pi f_c t + \theta)$$

where phase θ is uniformly distributed in $[-\pi, \pi]$

$$f_\theta(\theta) = \begin{cases} 1/2\pi & -\pi < \theta \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

$$R_X(\tau) = E[X(t+\tau)X(t)] = E\left[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \times \cos(2\pi f_c t + \theta)\right]$$

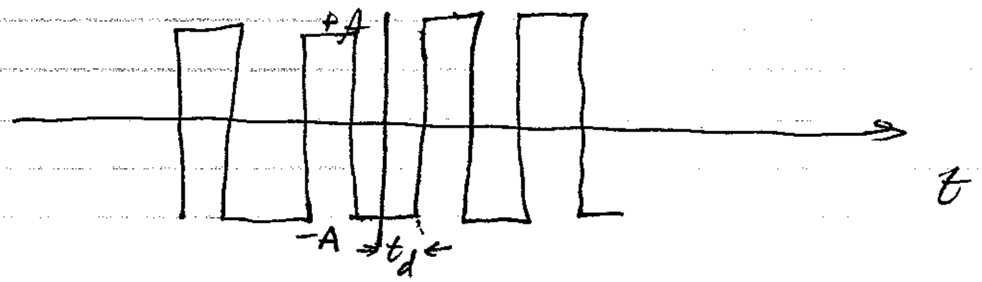
$$R_X(\tau) = \frac{A^2}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)] + \frac{A^2}{2} E[\cos(2\pi f_c \tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(2\pi f_c t + 2\pi f_c \tau + 2\theta)\,d\theta + \frac{A^2}{2}\cos(2\pi f_c \tau)$$

or

$$\boxed{R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)}$$

Ex: Random binary signal:



$$f_{T_d}(t_d) = \begin{cases} 1/T & 0 \le t_d \le T \\ 0 & \text{elsewhere} \end{cases}$$

a) if $|t_k - t_i| > T$ then

$$E[X(t_k)X(t_i)] = E[X(t_k)]E[X(t_i)] = 0$$

b) if $|t_k - t_i| < T$, then

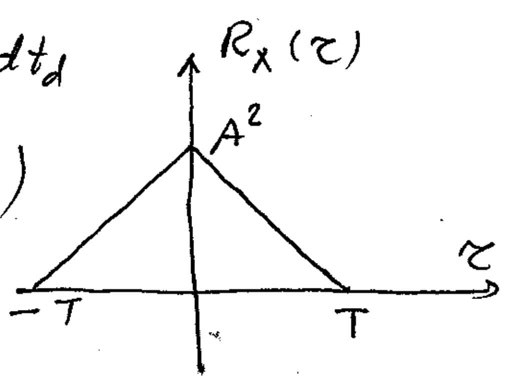
if $|t_k - t_i| < T - t_d$ then either we have $(-A)(-A)$ or $(+A)(+A)$, i.e., there is no new level. If, on the other hand $|t_k - t_i| > T - t_d \Rightarrow$ the two levels are independent and the average is zero.

$$E[X(t_k)X(t_i)|t_d] = \begin{cases} A^2 & t_d < T - |t_k - t_i| \\ 0 & \text{otherwise.} \end{cases}$$

or

$$E[X(t_k)X(t_i)] = \int_0^{T-|t_k-t_i|} A^2 f_{T_d}(t_d)\,dt_d$$

$$= A^2\left(1 - \frac{|t_k - t_i|}{T}\right)$$



Cross-Correlation

$$R_{XY}(t,u) = E[X(t)Y(u)]$$

if X and Y are WSS then

$$R_{XY}(t,u) = R_{XY}(\tau)$$

where $\tau = t - u$.

Transmission of a Random Process through a Linear filter:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

$$\mu_y(t) = E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, \mu_X(t-\tau_1)\, d\tau_1$$

for Wide Sense stationary Process $X(t)$

$$\mu_X(t) = \mu_X$$

So

$$\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 = \mu_X H(0) = \mu_Y$$

$$R_Y(t,u) = E[Y(t)Y(u)]$$

$$R_Y(t,u) = E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)\,d\tau_2\right.$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau_2)E[X(t-\tau_1)X(u-\tau_2)\,d\tau_2\right]h(\tau_1)\,d\tau_1$$

for WSS Processes:

$$R_Y(t,u) = R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1-\tau_2)\,d\tau_1\,d\tau_2$$

Power Spectral Density (PSD):

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\,d\tau$$

and

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}\,df$$

Properties of PSD:

1) $$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\,d\tau$$

2) $$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df$$

3) $S_X(f) \geq 0 \quad \forall f$

4) $S_X(-f) = S_X(f)$.

Example:

$$X(t) = A\cos(2\pi f_c t + \theta) \quad \text{where } \theta$$

is uniformly distributed over $[-\pi, \pi]$

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

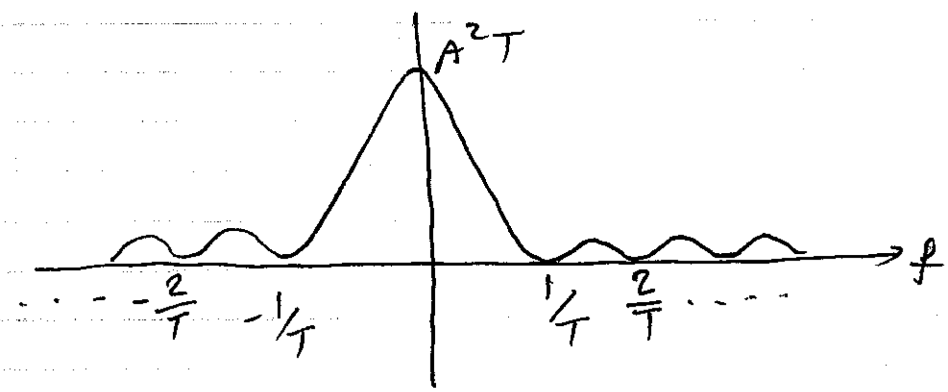
$$S_X(f) = \frac{A^2}{4}\left[\delta(f-f_c) + \delta(f+f_c)\right]$$

Example: Random binary pulse

$$R_X(\tau) = \begin{cases} A^2\left(1 - \frac{|\tau|}{T}\right) & |\tau| < T \\ 0 & |\tau| \geq T \end{cases}$$

$$S_X(f) = \int_{-T}^{T} A^2\left(1 - \frac{|\tau|}{T}\right) e^{-j2\pi f\tau}\, d\tau$$

$$S_X(f) = A^2 T \operatorname{sinc}^2(fT)$$



Relating PSD of input and output of linear Systems:

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_x(\tau - \tau_1 - \tau_2) e^{-j2\pi f \tau_1} d\tau_1 d\tau_2$$

or

$$S_y(f) = H(f) H^*(f) S_x(f)$$

or

$$S_y(f) = |H(f)|^2 S_x(f)$$

Gaussian random Processes:

We say a random variable has Gaussian distribution if its density function is given by,

$$f_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{\frac{-(y-\mu_y)^2}{2\sigma_y^2}}$$

where μ_y is the mean and σ_y^2 is the variance of the random variable.

When y is normalized, i.e., $\mu_y = 0$ and $\sigma_y^2 = 1$

we have

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Central limit theorem:

Take random variables $X_i, i = 1, 2, \dots, N$ such that

1) X_i's are statistically independent.
2) X_i's all have the same distribution

Then for the normalized version of $\{X_i\}$ as

$$Y_i = \frac{1}{\sigma_X}(X_i - \mu_X) \qquad i = 1, 2, \dots, N$$

Now

$$E[Y_i] = 0$$

and

$$Var[Y_i] = 1$$

Define $$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$$

the Central limit theorem says that V_N approaches a random variable with distribution $\mathcal{N}(0,1)$, i.e. a zero-mean, unit-variance Gaussian variable.

Properties of Gaussian Processes:

1) If a Gaussian random process is applied to a linear system, then the output is also Gaussian

2) Given mean and variance all higher order statistics of samples of a Gaussian random process can be characterized.

3) If a Gaussian Process is Wide Sense Stationa then it is also strict sense stationary.

4) The random variables $X(t_1), X(t_2), \ldots X(t_n)$ obtained by sampling ~~that~~ a Gaussian Process $X(t)$ are uncorrelated

$$E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})] = 0 \qquad i \neq k$$

in case of Gaussian process, these samples are also statistically independent.

Thermal Noise

$$E[V_{TN}^2] = 4kTRB$$

where $k = 1.38\times10^{-23}$ is the Boltzman's Constant, B is the bandwidth and R is the resistance.

Maximum Power that can be delivered is to a matched load of R Ω, i.e.

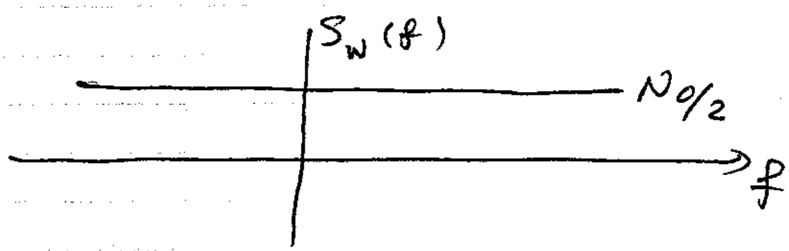
$$N = \left(\frac{\sqrt{4kTRB}}{R+R}\right)^2 R = kTB$$

if we assume that noise is uniformly distributed over all frequencies

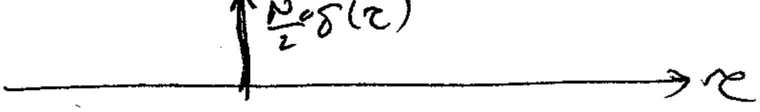
$$N_o = \frac{N}{B} = kT$$

This is the starting point in defining the white noise, i.e., a process W with

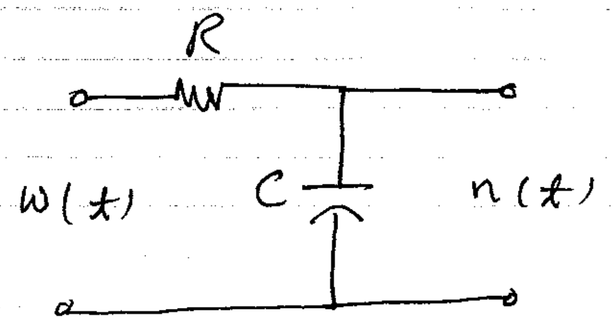
$$S_W(f) = \frac{N_o}{2} \qquad \forall f$$



$$R_W(\tau) = \mathcal{F}^{-1}\left[\frac{N_o}{2}\right] = \frac{N_o}{2}\delta(\tau)$$



Example: RC low pass filtered White Noise



$$H(f) = \frac{1}{1 + j2\pi f RC}$$

$$|H(f)|^2 = H(f)H^*(f) = \frac{1}{1 + (2\pi f RC)^2}$$

$$S_N(f) = S_W(f)|H(f)|^2 = \frac{N_o/2}{1 + (2\pi f RC)^2}$$

$$R_N(\tau) = \frac{N_o}{4RC} e^{-\frac{|\tau|}{RC}}$$

where we have used

$$e^{-a|\tau|} \Longleftrightarrow \frac{2a}{a^2 + (2\pi f)^2}$$