Lecture 4

Baseband Transmission

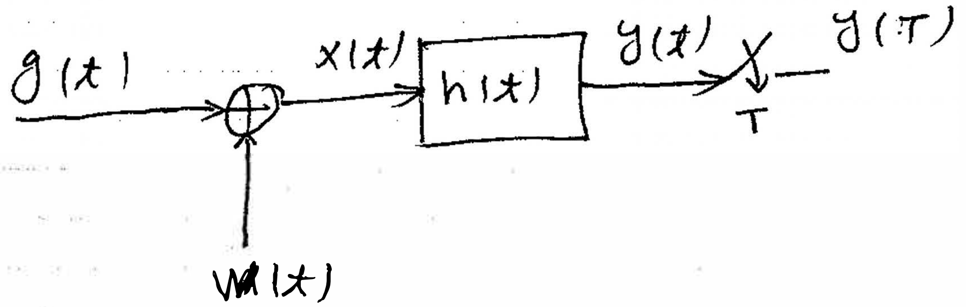
Matched Filter

Assume that a signal $g(t)$ is transmitted in AWGN (Additive White Gaussian Noise) channel.

Then the received signal is

$$x(t) = g(t) + w(t) \qquad 0 \leq t \leq T$$

where T is the signalling interval, i.e., the duration of a signal representing one or more bits.



The received (noisy) signal goes through a receiver filter with impulse response $h(t)$. The output is

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

or

$$y(t) = g_0(t) + n(t)$$

where

$$g_0(t) = \int_{-\infty}^{\infty} g(t-\tau)\,h(\tau)\,d\tau$$

and

$$n(t) = \int_{-\infty}^{\infty} w(t-\tau)\,h(\tau)\,d\tau$$

Our goal is to maximize the Signal-to-Noise ratio at the output of the receiver filter, i.e., to maximize:

$$\eta = \frac{|g_0(T)|^2}{E[n^2(t)]}$$

$$g_0(t) = \mathcal{F}^{-1}[G(f)] = \mathcal{F}^{-1}[H(f)\,G(f)]$$

$$= \int_{-\infty}^{\infty} H(f)\,G(f)\,e^{j2\pi f t}\,df$$

So

$$|g_0(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)\,G(f)\,e^{j2\pi f T}\,df \right|^2$$

and $$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\,df$$

But $S_N(f) = \frac{N_0}{2}|H(f)|^2$

So:

$$\eta = \frac{\left|\int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi f T} df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df}$$

We can use Schwarz inequality to find the maximum of η.

<u>Schwarz's inequality</u>

let $\phi_1(x)$ and $\phi_2(x)$ be two functions such that

$$\int_{-\infty}^{\infty}|\phi_1(x)|^2 dx < \infty$$

and

$$\int_{-\infty}^{\infty}|\phi_2(x)|^2 dx < \infty$$

then

$$\left|\int_{-\infty}^{\infty}\phi_1(x)\phi_2(x)dx\right|^2 \leq \int_{-\infty}^{\infty}|\phi_1(x)|^2 dx \int_{-\infty}^{\infty}|\phi_2(x)|^2 dx$$

with equality if

$$\phi_1(x) = k\,\phi_2^*(x).$$

So:

$$\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi f T} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |G(f)|^2 df$$

So:

$$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df$$

with equality if

$$H(f) = H_{opt}(f) = k\, G^*(f)\, e^{-j2\pi f T}$$

or

$$h_{opt}(t) = k \int_{-\infty}^{\infty} G^*(f)\, e^{-j2\pi f T} \cdot e^{j2\pi f t}\, df$$

if $g(t)$ is real

$$G^*(f) = G(-f)$$

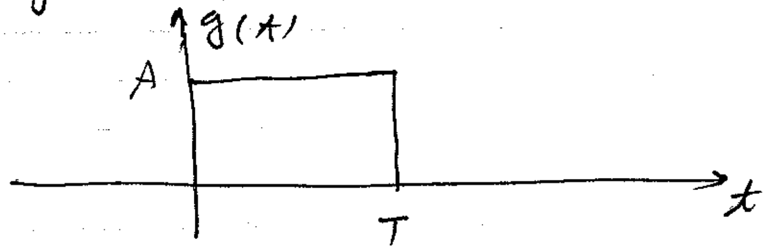
and

$$h_{opt}(t) = k \int_{-\infty}^{\infty} G(-f)\, e^{-j2\pi f (T-t)}\, df$$

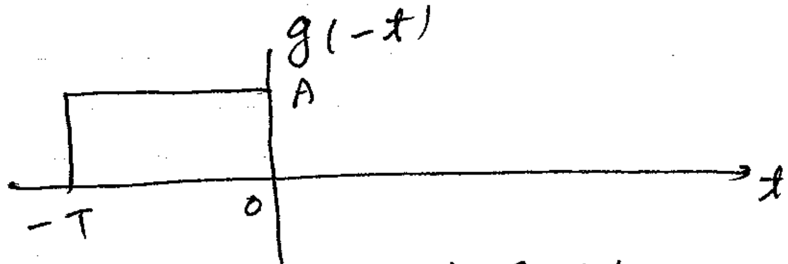
$$= k\, g(T-t)$$

This is called a filter matched to $g(t)$ (a <u>matched filter</u>). This is a <u>time-reversed</u> and shifted version of $g(t)$.

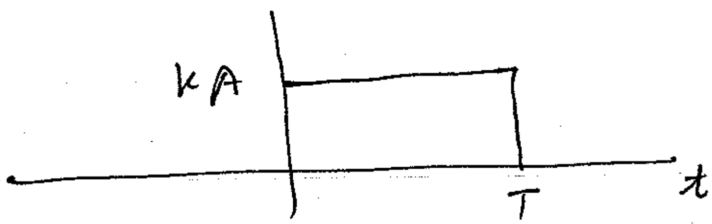
Example: The matched filter for a rectangular pulse



$$h(t) = kg(T-t)$$

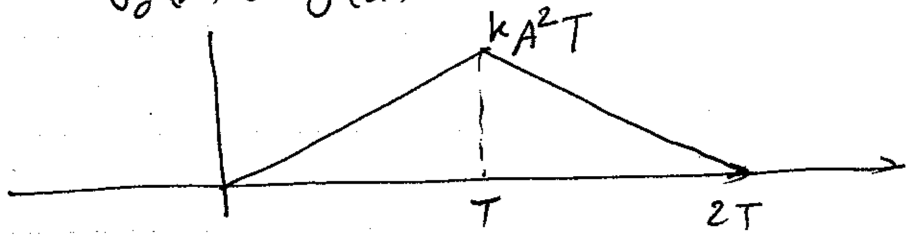


$$g(T-t) = g(t)$$



Matched filter output

$$g_o(t) = g(t) * h(t)$$



Note that

$$\eta_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2 df = 2\frac{E}{N_0}$$

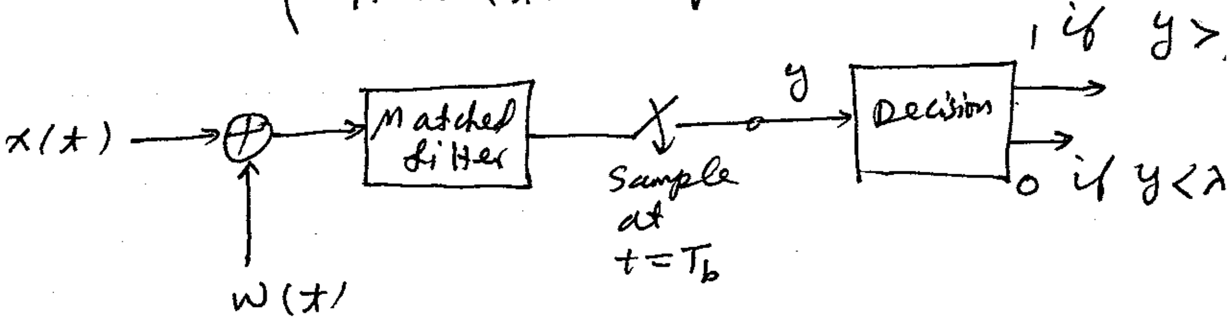
where

$$E = \int_{-\infty}^{\infty} |G(f)|^2 df$$

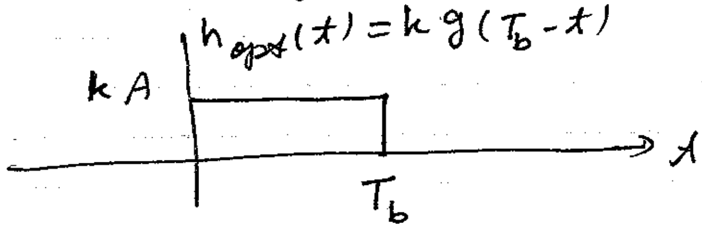
Binary Detection : Probability of error

Assume that we would like to transmit a set of binary values, a stream of bits, over an AWGN channel. Every T_b seconds we transmit a pulse of amplitude A if we have a 1 and a pulse of amplitude of $-A$ to represent a zero. So,

$$x(t) = \begin{cases} A + w(t) & \text{if the bit is 1} \\ -A + w(t) & \text{if the bit is 0.} \end{cases}$$



The matched filter is



let $kAT_b = 1$ (This is just for convenience and does not have any physical significance)

Then

$$y = \int_0^{T_b} x(t)\,dt = \pm A + \frac{1}{T_b}\int_0^{T_b} w(t)\,dt$$

y has the mean $\pm A$ depending on whether 0 or 1 was send. for $\bar{Y} = -A$ (when 0 sent)

$$\sigma_Y^2 = E\left[(Y+A)^2\right] = \frac{1}{T_b^2} E\left[\int_0^{T_b}\int_0^{T_b} w(t)\,w(u)\,dt\,du\right]$$

$$= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} E[w(t)w(u)]\,dt\,du$$

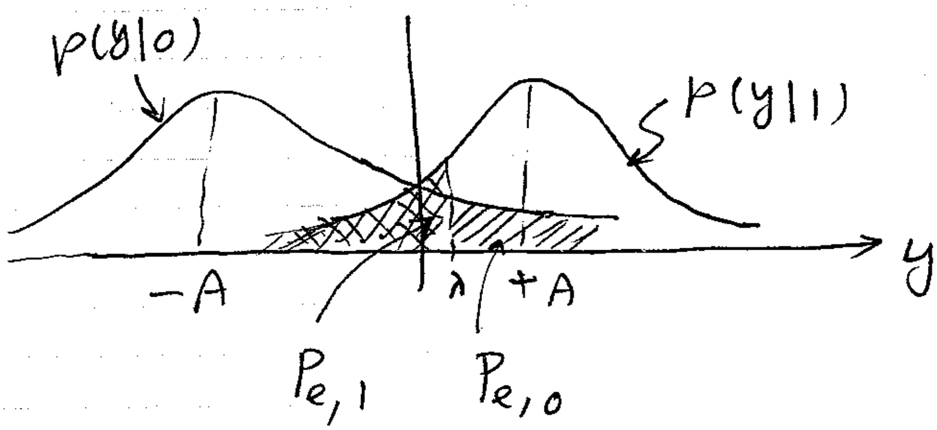
$$= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\,\delta(t-u)\,dt\,du = \frac{N_0}{2T_b}$$

So,

$$f_Y(y|0) = \frac{1}{\sqrt{\frac{\pi N_0}{T_b}}} \exp\left[-\frac{(y+A)^2}{N_0/T_b}\right]$$

and

$$f_Y(y|1) = \frac{1}{\sqrt{\frac{\pi N_0}{T_b}}} \exp\left[-\frac{(y-A)^2}{N_0/T_b}\right]$$



$$P_{e,0} = P[\text{error} \mid 0 \text{ sent}]$$
$$= \int_{\lambda}^{\infty} f_y(y|0)\, dy$$

$$P_{e,1} = P[\text{error} \mid 1 \text{ sent}]$$
$$= \int_{-\infty}^{\lambda} f_y(y|1)\, dy$$

overall Probability of error

$$P_e = P_0 P_{e,0} + P_1 P_{e,1}$$

where p_0 is the probability that a 0 was sent. If $P_0 = P_1 = 1/2$ then it is reasonable to take $\lambda = 0$ and

$$P_e = \frac{1}{2} P_{e,0} + \frac{1}{2} P_{e,1} = P_{e,0} = P_{e,1}$$

We have

$$P_{e,0} = \int_0^\infty \frac{1}{\sqrt{\frac{\pi N_0}{T_b}}} e^{-\frac{(y+A)^2}{N_0/T_b}} dy$$

Let $\quad z = \frac{(y+A)}{\sqrt{N_0/2T_b}}$

then $\quad dz = \frac{dy}{\sqrt{\frac{N_0}{2T_b}}}$

and

$$P_{e,0} = \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{\frac{N_0}{2T_b}}}}^\infty e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{A\sqrt{2T_b}}{\sqrt{N_0}}}^\infty e^{-z^2/2} dz$$

$A^2 T_b = E_b$ energy per bit so:

$$P_{e,0} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E_b}{N_0}}}^\infty e^{-z^2/2} dz = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-z^2/2} dz.$$

So, Probability of error is:

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_o}}\right)$$

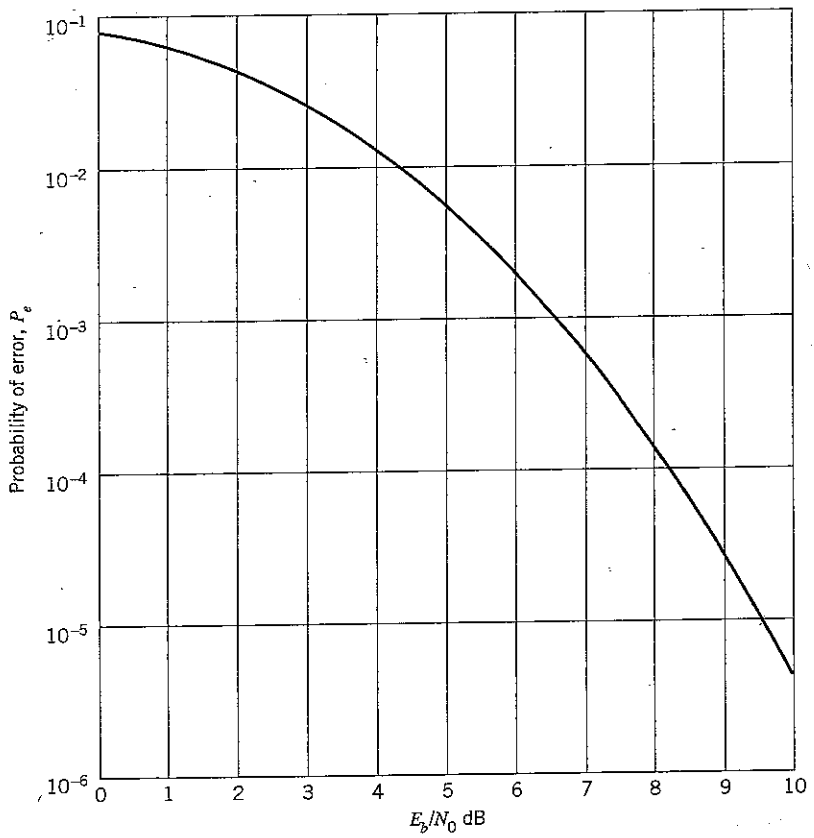


FIGURE 8.7 Probability of error in additive white Gaussian noise with binary signaling.