Review of Signals and Systems

Fourier Transform of a time function (a signal) $g(t)$ is defined as:

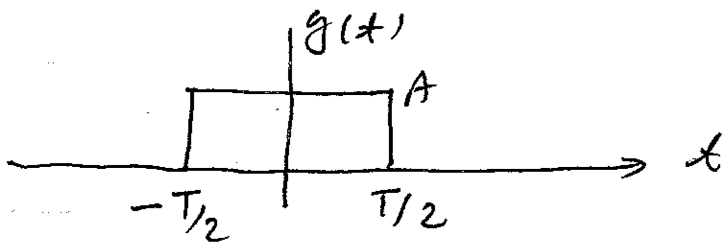
$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

$g(t)$ can be recovered from $G(f)$ as:

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

$g(t)$ is called the inverse Fourier Transform of $G(f)$.

<u>Example</u>: A rectangular pulse



$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$$

where

$$\mathrm{rect}(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & |t| \geq \frac{1}{2} \end{cases}$$

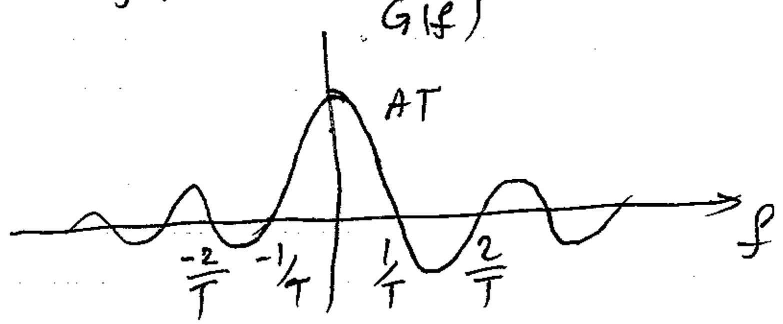
$$G(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi f t}\, dt$$

$$= \frac{A}{-j2\pi f}\, e^{-j2\pi f t}\Big|_{-T/2}^{T/2}$$

$$= A\, \frac{-e^{j2\pi f T/2} - e^{-j2\pi f T/2}}{-j2\pi f}$$

$$= A\, \frac{e^{j\pi f T} - e^{-j\pi f T}}{j2} \times \frac{1}{\pi f} = \frac{A \sin(\pi f T)}{\pi f}$$

$$= AT\, \frac{\sin(\pi f T)}{\pi f T} = AT\, \mathrm{Sinc}(fT)$$

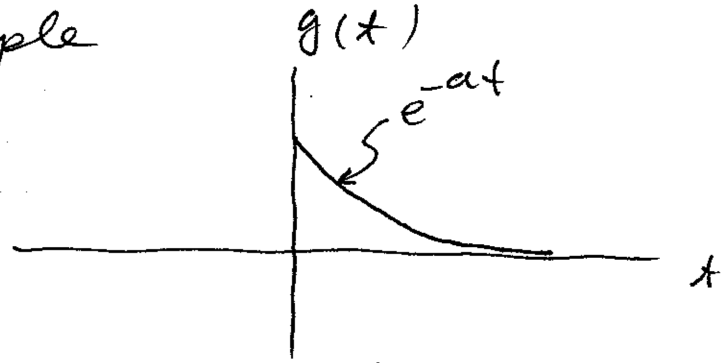


So :

$$A\, \mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow AT\, \mathrm{Sinc}(fT)$$

Note that as T increases the pulse expands in time domain while its Fourier Transform Compresses in frequency domain.

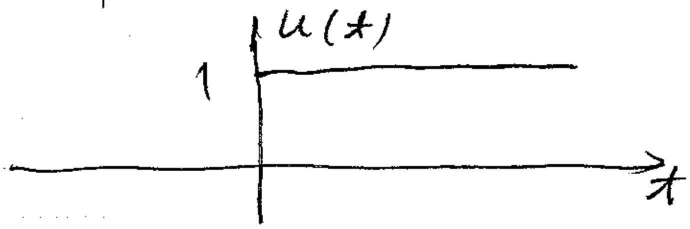
Example



$$g_1(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

define the unit step function as

$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$



then

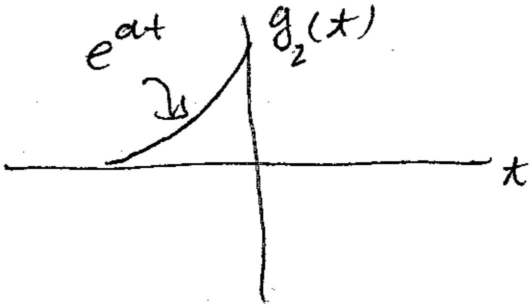
$$g_1(t) = e^{-at}\, u(t)$$

$$G_1(f) = \int_0^{\infty} e^{-at} \cdot e^{-j2\pi f t}\, dt$$

$$= \frac{1}{a+j2\pi f}$$

$$e^{-at}\, u(t) \Longleftrightarrow \frac{1}{a+j2\pi f}$$

Now Consider:

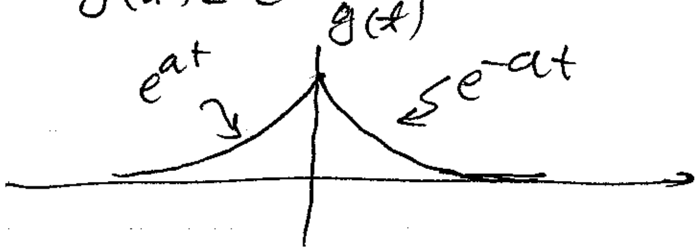


$$g_2(t) = e^{at}\, u(-t)$$

$$G_2(f) = \int_{-\infty}^{\infty} e^{at} e^{-j2\pi f t}\, dt = \frac{1}{a - j2\pi f}$$

Example:

$$g(t) = e^{-a|t|}$$



$$G(f) = \mathcal{F}\left[g_1(t) + g_2(t)\right] = G_1(f) + G_2(f)$$

$$= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

So:

$$e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2 + (2\pi f)^2}$$

In the above we use the linear property of the Fourier Transform. That is,

$$\mathcal{F}[a_1 g_1(t) + a_2 g_2(t)] = a_1 G_1(f) + a_2 G_2(f)$$

Time scaling property

$$g(at) \Leftrightarrow \frac{1}{|a|} G(f/a)$$

where $G(f)$ is the Fourier Transform of $g(t)$.

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j2\pi f t} dt$$

let $\tau = at$ then

$$\mathcal{F}[g(at)] = \frac{1}{a}\int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi \frac{f}{a}\tau} d\tau$$

$$= \frac{1}{a} G\left(\frac{f}{a}\right) \quad \text{if } a > 0$$

$$-\frac{1}{a} G\left(\frac{f}{a}\right) \quad \text{if } a < 0$$

So

$$\mathcal{F}[g(at)] = \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

let $a = -1$ then

$$g(-t) \Leftrightarrow G(-f)$$

Duality Property:

if $g(t) \Leftrightarrow G(f)$

then

$$G(t) \Leftrightarrow g(-f)$$

Proof:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

change t and f :

$$g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi f t} dt$$

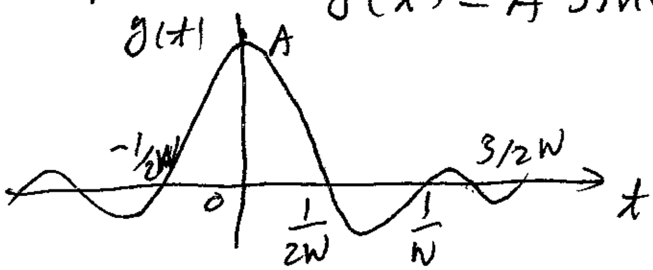
let $f \to -f$

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t} dt$$

but the above integral is the Fourier Transform of $G(t)$. So,

$$G(t) \Leftrightarrow g(-f).$$

example: $g(t) = A\,\mathrm{Sinc}(2Wt)$



we know that

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow AT\,\mathrm{Sinc}(fT)$$

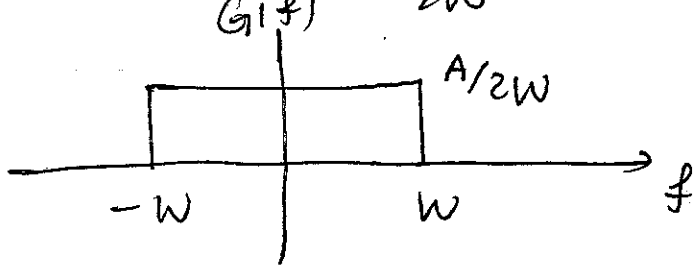
let $T = 2W$

$$A\,\mathrm{rect}\left(\frac{t}{2W}\right) \Leftrightarrow A2W\,\mathrm{Sinc}(2fW)$$

$$\frac{A}{2W}\mathrm{rect}\left(\frac{t}{2W}\right) \Leftrightarrow A\,\mathrm{Sinc}(2fW)$$

Using duality property:

$$A\,\mathrm{Sin}(2tW) \Leftrightarrow \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$



Time shifting property

if $g(t) \Leftrightarrow G(f)$

$$g(t - t_0) \Leftrightarrow G(e)\, e^{-j2\pi f t_0}$$

Proof:

$$\mathcal{F}[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)\, e^{-j2\pi f t}\, dt$$

let $t - t_0 = \tau$

Then

$$\mathcal{F}[g(t-t_0)] = e^{-j2\pi f t_0}\int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$= e^{-j2\pi f t_0}\, G(f)$$

Frequency Shift

$$\exp(j2\pi f_c t)\, g(t) \Leftrightarrow G(f - f_0)$$

Proof

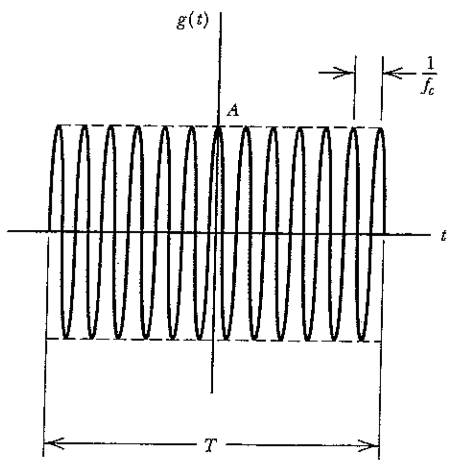
$$\mathcal{F}[e^{j2\pi f_c t} g(t)] = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi t (f - f_c)}\, dt$$

$$= G(f - f_c)$$

Example:

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) \cos 2\pi f_c t$$

this is like modulating a rectangular pulse:



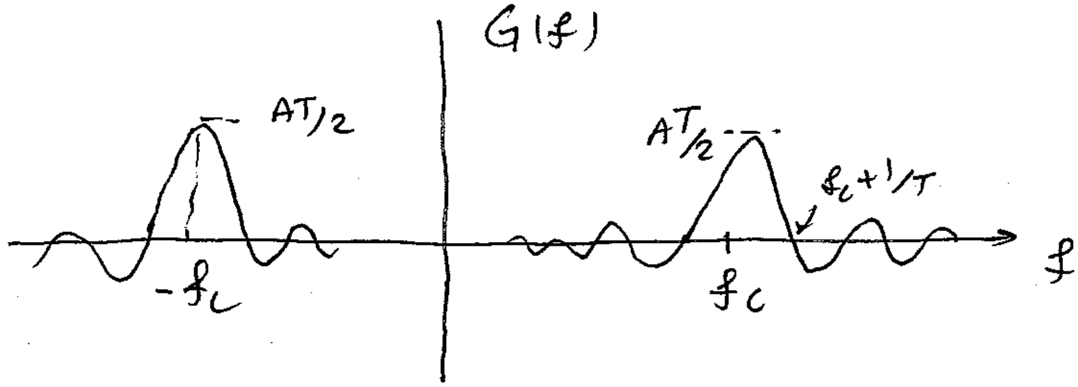
(a)

Note that

$$\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$$

So

$$G(f) = \frac{AT}{2}\left\{ \operatorname{Sinc}[(f-f_c)T] + \operatorname{Sinc}[(f+f_c)T] \right\}$$



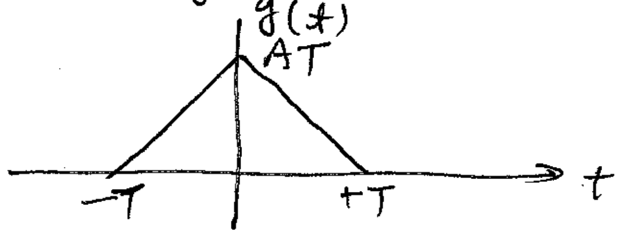
differentiation in time domain

$$\frac{d}{dt} g(t) = j2\pi f \; G(f)$$

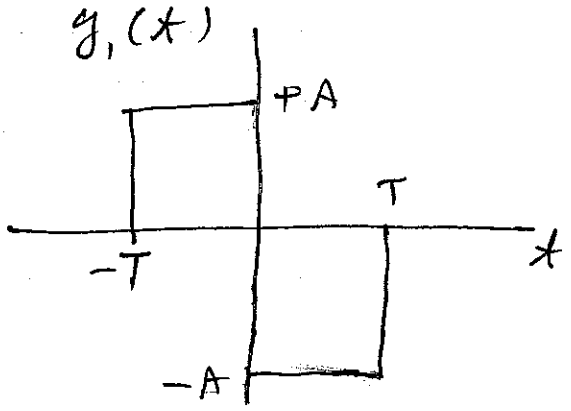
integration in time domain:

$$\int_{-\infty}^{t} g(\tau)\, d\tau \Leftrightarrow \frac{1}{j2\pi f} G(f)$$

Example: triangular pulse



The derivative of $g(t)$ is



$$G_1(f) = AT\,\text{sinc}(fT)\,e^{j2\pi f T/2} - AT\,\text{sinc}(fT)\,e^{-j2\pi f T/2}$$

$$= 2jAT\,\text{sinc}(fT)\,\sin(\pi f T)$$

$$G(f) = \frac{1}{j2\pi f} G_1(f) = AT\,\text{sinc}(fT)\,\frac{\sin(\pi f T)}{\pi f}$$

$$= AT^2\,\text{sinc}(fT)\,\text{sinc}(fT)$$

$$= AT^2\,\text{sinc}^2(fT)$$

Convolution in time domain

$$\int g_1(\tau)\, g_2(t-\tau)\, d\tau \Leftrightarrow G_1(f)\, G_2(f)$$

Rayleigh energy Theorem:

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

where $|g(t)|^2 = g(t)\, g^*(t)$

and $|G(f)|^2 = G(f)\, G^*(f)$

Delta Function:

Dirac delta function $\delta(t)$ is defined as:

$$\delta(t) = 0 \qquad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

if we multiply a function $g(t)$ with $\delta(t - t_0)$ and integrate, we get

$$\int_{-\infty}^{\infty} g(t)\, \delta(t - t_0)\, dt = g(t_0)$$

letting $\tau = t$ and $t = t_0$ we get:

$$\int_{-\infty}^{\infty} g(\tau)\, \delta(t - \tau)\, d\tau = g(t)$$

that is, every function $g(t)$ can be written as a weighted sum (integral) of $\delta(\cdot)$ functions.

Fourier Transform of $\delta(t)$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$

So

$$\delta(t) \Longleftrightarrow 1$$

using time-shift property:

$$\delta(t - t_0) \Longleftrightarrow e^{-j2\pi f t_0}$$

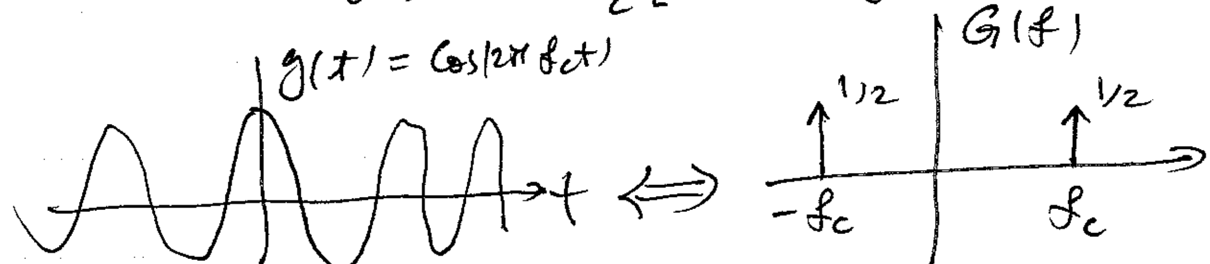
Similarly

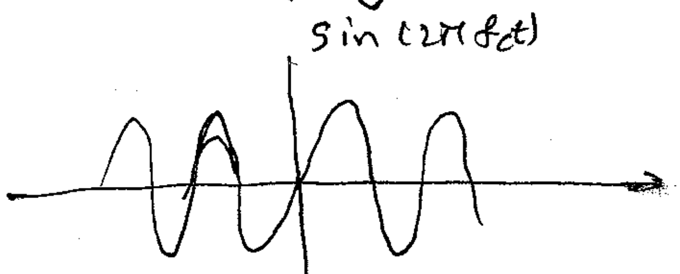
$$e^{j2\pi f_c t} \Longleftrightarrow \delta(f - f_0)$$

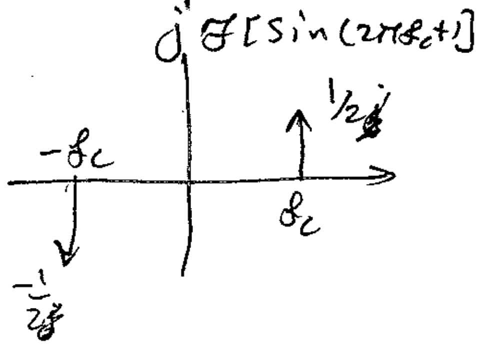
$$\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$$

So,

$$\cos(2\pi f_c t) \Longleftrightarrow \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$







$$\sin(2\pi f_c t) = \frac{1}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right]$$

So

$$\sin(2\pi f_c t) \Longleftrightarrow \frac{1}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right]$$

Transmission of Signals through linear systems:

a System is linear if

$$L\left[a x_1(t) + b x_2(t)\right] = a L\left[x_1(t)\right] + b L\left[x_2(t)\right]$$

This idea can be generalized (by induction) ~~to a weighted sum of (or~~ weighted integral of) functions.

For an input $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

we have:

$$y(t) = L\left[x(t)\right] = L\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right]$$

or

$$y(t) = \int_{-\infty}^{\infty} x(\tau) L[\delta(t-\tau)]\, d\tau$$

So, in order to be able to find the out-put of a linear-time-invariant (LTI) system to any function, we only need to know its response to a delta function

let $\quad L[\delta(t)] = h(t)$

That is when $x(t) = \delta(t)$, $y(t) = h(t)$.

$h(t)$ is called the <u>impulse</u> <u>response</u> of the system.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$= x(t) \circledast h(t)$$

where $\circledast$ represent Convolution.

Taking Fourier Transform of both sides:

$$Y(f) = X(f)\, H(f)$$

where $Y(f)$, $X(f)$ and $H(f)$ are the Fourier Transforms of $y(t)$, $x(t)$ and $h(t)$.

$H(f)$ is called the transfer function:

$$H(f) = \frac{Y(f)}{X(f)}.$$