ELEC 6841

Digital Communications II

Lecture 1, Jan. 7, 2010

Carrier & Symbol Synchronization

- Importance of Synchronization in a Communication System.
- Different Synchronization tasks:
  - Phase Synchronization: Carrier Recovery.
  - Bit (Symbol) Synchronizations.
  - Frame Synchronization (preamble detection, etc
  - Higher level Synchronization.
- In this lecture, we only discuss Carrier Recovery and Symbol Recovery.

The model:

Assume that a signal $s(t)$ is transmitted.

$$s(t) \longrightarrow s(t-\tau)+n(t)$$

The receiver, gets:

$$r(t) = s(t-\tau)+n(t)$$

where $n(t)$ is the noise and τ is the delay.

$s(t)$ is the information carrying signal. In the baseband

$$s(t) = s_m(t) \qquad m \in \{0, 1, \cdots, M-1\}$$

e.g., in the case of binary ($M=2$), we have:

$$s(t) = s_0(t) \quad \text{or} \quad s(t) = s_1(t)$$

For the carrier-modulated (band-pass) case:

$$s(t) = s_m(t)\cos(2\pi f_c t) = Re[s_{m\ell}(t) e^{j2\pi f_c t}]$$

where $s_{m\ell}(t)$ is the low-pass equivalent signal of $s_m(t)$.

So,

$$r(t) = Re[s_\ell(t-\tau) e^{j2\pi f_c (t-\tau)}] + n(t)$$

or

$$r(t) = Re\{[s_\ell(t-\tau)e^{j\phi} + z(t)] e^{j2\pi f_c t}\}$$

where $\phi = -2\pi f_c \tau$ is the carrier phase due to propagation delay τ.

If this phase difference was the only source of phase discrepancy, then, we only needed to find (estimate) τ and find ϕ based on it.

However, due to the fact that 1) the phase difference due to the drift between transmitter and receiver oscillators should also be added and 2) the accuracy of τ is not sufficient for proper calculation of ϕ, we need to consider phase recovery (estimation of ϕ) and carrier recovery (estimation of τ) as two different tasks.

As an example assume that a system transmits at a rate of 2 Mbps at a frequency of 5 GHz. The bit duration is $\frac{1}{2\times10^{6}} = 5\times10^{-7}$ sec. or 0.5 μsec. If we have an accuracy of better than $\frac{1}{20}$ th of bit interval in timing recovery (bit synchronization) we observe no noticable degradation in BER.

So, we are content with $\frac{0.5}{20} = 0.025\ \mu\text{sec} = 25$ nsec

But this results in

$$\Delta\phi = 2\pi\times5\times10^{9}\times25\times10^{-9} = 250\pi$$

uncertainty in phase estimation!!

So, we represent $r(t)$ as:

$$r(t) = s(t;\phi,\tau) + n(t)$$

There are many ways to estimate a parameter (say, phase, delay τ, etc.). But, they fall into two basic schemes (and their approximations). These are:

- MAP: Maximum A posteriori Probability.

and

ML: Maximum Likelihood.

- In the MAP estimation technique, the parameter to be detected (θ) is assumed to be a random variable (a random vector, in general) with apriori probability density $p(\theta)$.
- In the ML technique θ is treated as an <u>un-known</u>, but, <u>deterministic</u> parameter.

Let's represent $r(t)$ by a vector

$$\underline{r} = (r_1, r_2, \dots, r_N)$$

where r_i, $i = 1, \dots, N$ is the projection of $r(t)$ along $\phi_j(t) \in \{\phi_n(t)\}$.

The Conditional pdf of $\underline{r}$ (given $\underline{\theta}$) is $p(\underline{r}|\underline{\theta})$

The ML estimate of $\underline{\theta}$ is the value of $\underline{\theta}$ that maximizes $p(\underline{r}|\underline{\theta})$.

The MAP estimate is the value of $\underline{\theta}$ that maximizes

$$p(\underline{\theta}|\underline{r}) = \frac{p(\underline{\theta})\,p(\underline{r}|\underline{\theta})}{p(\underline{r})} .$$

Note that, when there is no a priori information about $\underline{\theta}$, we can assume $p(\underline{\theta})$ to be uniform. Then MAP will become ML.

Assuming that the noise is AWGN:

$$p(\underline{r}|\underline{\theta}) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \exp\left[-\sum_{n=1}^{N} \frac{(r_n - s_n(\underline{\theta}))^2}{2\sigma^2}\right]$$

where:

$$r_n = \int_{T_0} r(t)\,\phi_n(t)\,dt$$

$$s_n(\underline{\theta}) = \int_{T_0} s(t;\underline{\theta})\,\phi_n(t)\,dt .$$

where T_0 is the integration interval.

as $N \to \infty$:

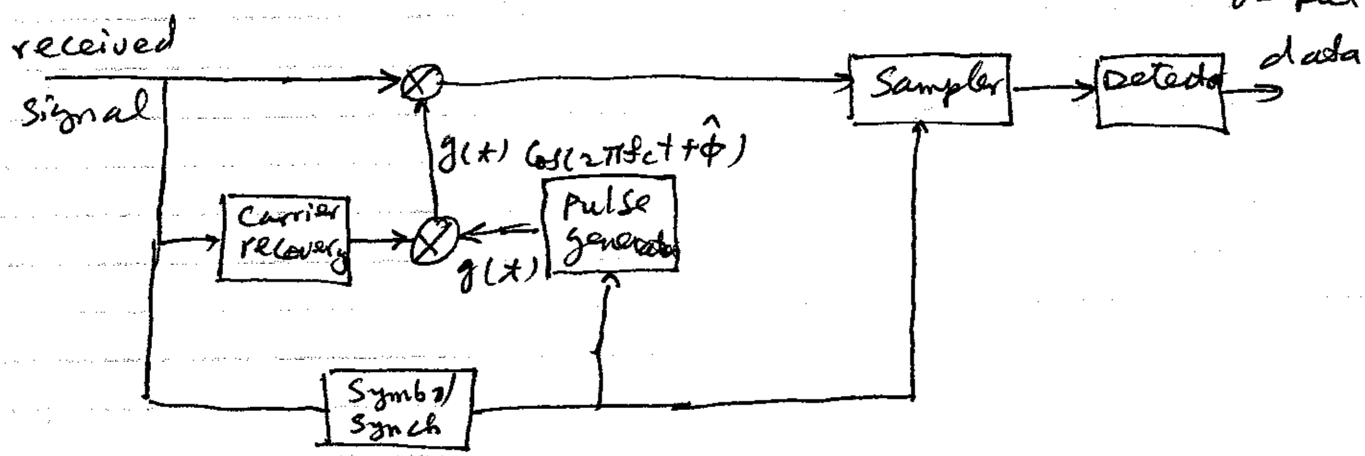
$$\lim_{N\to\infty} \frac{1}{2\sigma^2} \sum_{n=1}^{\infty} [r_n - s_n(\underline{\theta})]^2 = \frac{1}{N_0} \int_{T_0} [r(t) - s(t;\underline{\theta})]^2\,dt$$

Given the fact that $\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N$ is constant, maximization of $p(\underline{r}|\underline{\theta})$ is equivalent to maximization of the likelihood function:

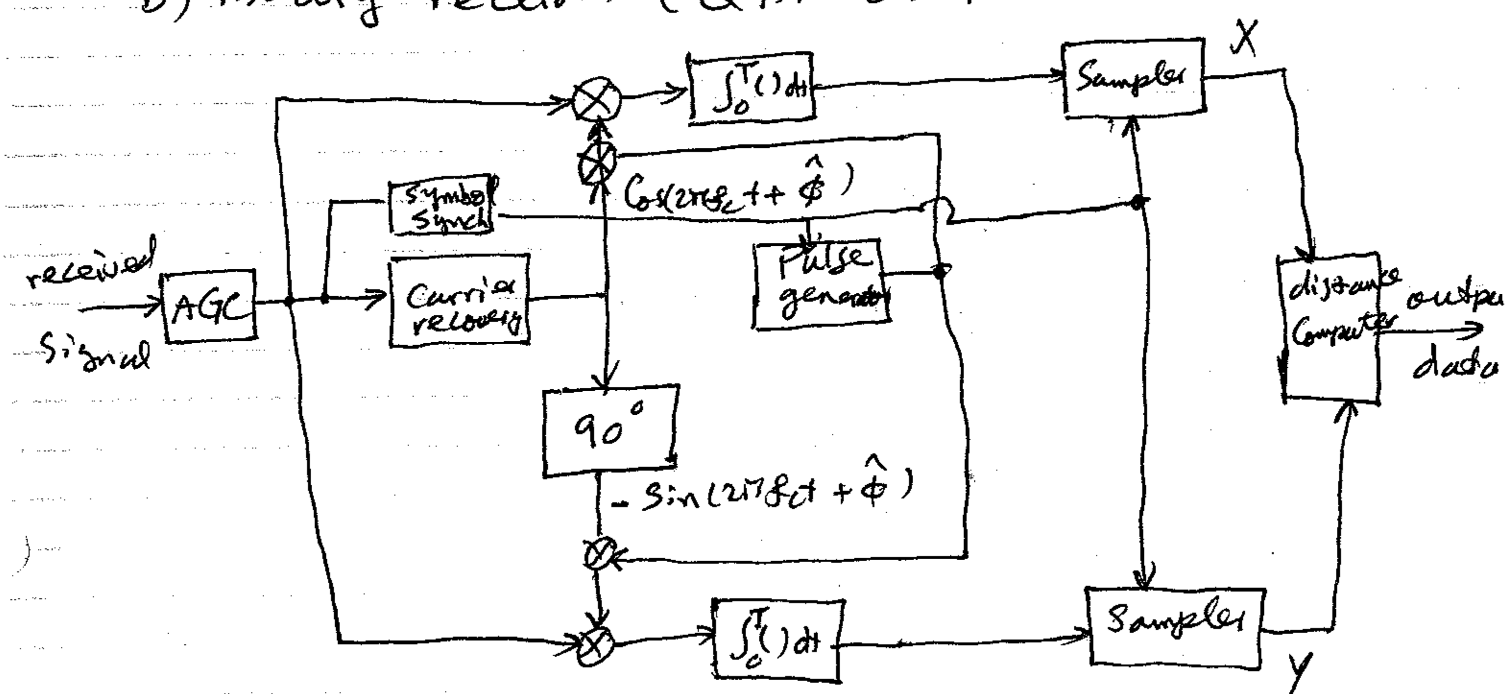
$$\Lambda(\underline{\theta}) = \exp\left[-\frac{1}{N_0}\int_{T_0}[r(t)-s(t;\theta)]^2\,dt\right]$$

The block Diagram of a receiver:

a) Binary receiver (e.g., BPSK):



b) M-ary receiver (QAM or MPSK):



Carrier Recovery (Phase recovery)

Assume that $s(t) = A(t)\cos(2\pi f_c t + \phi)$

has been transmitted and the receiver's

oscillator generates $c(t) = \cos(2\pi f_c t + \hat{\phi})$

as a replica of the carrier. Then

$$c(t)s(t) = \frac{1}{2}A(t)\cos(\phi - \hat{\phi}) + \frac{1}{2}A(t)\cos(4\pi f_c t + \phi + \hat{\phi})$$

After integrating (low-pass filtering), we have

$$y(t) = \frac{1}{2}A(t)\cos(\phi - \hat{\phi})$$

Note the effect of phase error in signal power

The power is degraded (SNR or E_b/N_0 reduced)

by a factor of $\cos^2(\phi - \hat{\phi})$. So for 10°

phase error there is 0.13 dB SNR degradation.

For 30° it is 1.25 dB. In QAM the effect

is even worse as it results in cross-talk as well.

Let $s(t) = A(t)\cos(2\pi f_c t + \phi) - B(t)\sin(2\pi f_c t + \phi)$

and $c_I(t) = \cos(2\pi f_c t + \hat{\phi})$,

$c_q(t) = -\sin(2\pi f_c t + \hat{\phi})$

Then

$$y_I(t) = \frac{1}{2}A(t)\cos(\theta - \hat{\theta}) - \frac{1}{2}B(t)\sin(\theta - \hat{\theta})$$

$$y_Q(t) = \frac{1}{2}B(t)\cos(\theta - \hat{\theta}) + \frac{1}{2}A(t)\sin(\theta - \hat{\theta}).$$

Maximum-Likelihood phase estimation:

In this case $\theta = \phi$ and the likelihood function is:

$$\Lambda(\phi) = \exp\left[-\frac{1}{N_0}\int_{T_0}[r(t) - s(t;\phi)]^2 dt\right]$$

Note that maximizing is equivalent to minimizing

$$d(r(t), \phi) = \frac{1}{N_0}\int_{T_0}[r(t) - s(t;\phi)]^2 dt$$

i.e., to minimize the distance between $r(t)$ an $s(t;\phi)$

we can write

$$d(r(t), \phi) = \frac{1}{N_0}\int_{T_0} r^2(t)\, dt - \frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)dt + \frac{1}{N_0}\int_{T_0} s^2(t;\phi)dt$$

The first and last term are constant. So, the problem of phase estimation consists in maximizing

$$\Lambda_L(\phi) = \frac{2}{N_0}\int_{T_0} r(t)\, s(t;\phi)dt$$

This means finding the waveform $s(t;\phi)$ that has the most (highest value) of correlation with $r(t)$.

Example:

$$r(t) = A\cos(2\pi f_c t + \phi) + n(t)$$

$$\Lambda_L(\phi) = \frac{2A}{N_0}\int_{T_0}\cos(2\pi f_c t + \phi)\, r(t)dt$$

To maximize $\Lambda_L(\phi)$ w.r.t. ϕ, we need to take its derivative and equate it to zero:

$$\frac{d\,\Lambda_L(\phi)}{d\,\phi} = 0$$

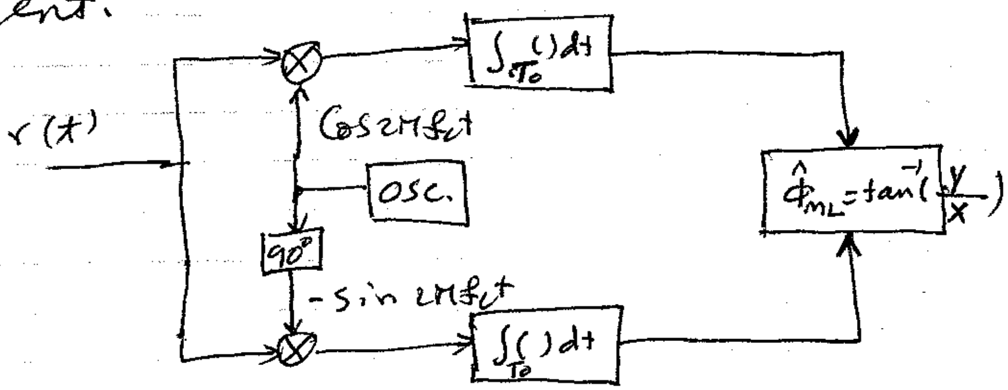
or

$$\int_{T_0} r(t)\,\sin(2\pi f_c t + \hat{\phi}_{ML})\,dt = 0$$

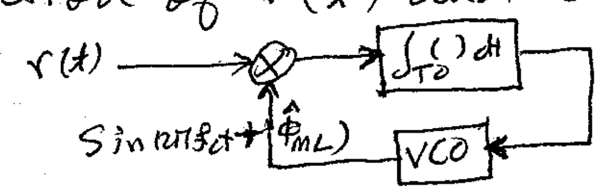
This gives us:

$$\hat{\phi}_{ML} = -\tan^{-1}\left[\frac{\int_{T_0} r(t)\sin 2\pi f_c t\,dt}{\int_{T_0} r(t)\cos 2\pi f_c t\,dt}\right].$$

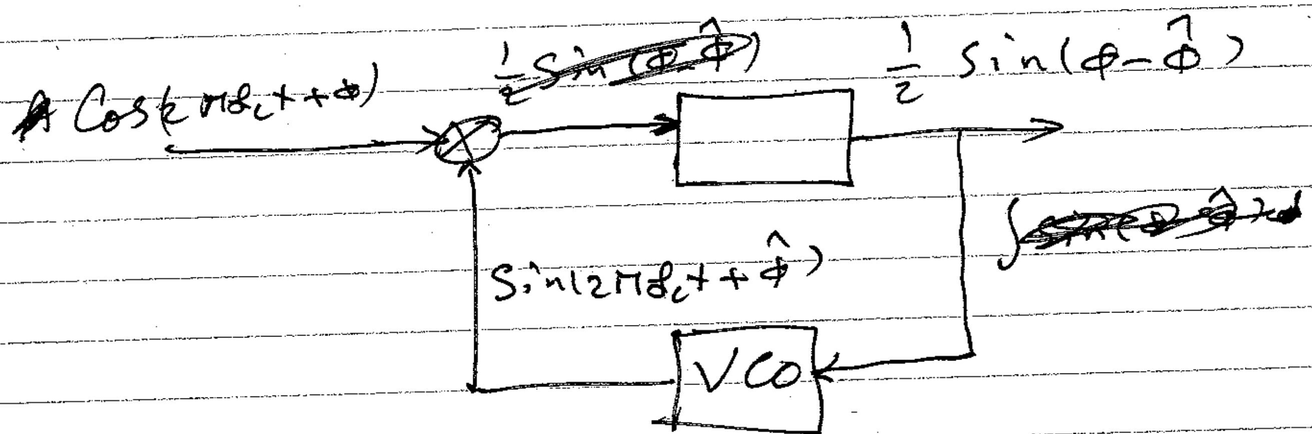
This means that we need to divide the sum of quadrature phase components by the sum of in-phase components and then take the inverse tangent.



Another way is to use a loop that tries to enforce the correlation of $r(t)$ and $S(2\pi f_c t + \hat{\phi}_{ML})$ to zero:



$\sin(\phi - \hat{\phi})$



$$f_i(t) = f_c + k\int_0^t e(t')\,dt'$$

$$f_i(t) = f_c + K\,v(t)$$

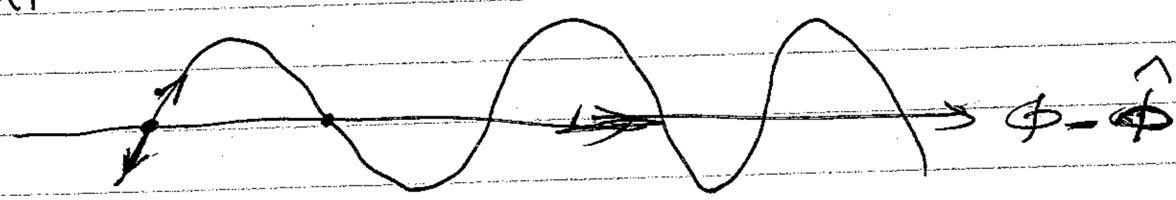
$$\hat{\phi}(t) = 2\pi f_c t + K\int_{-\infty}^{t} v(t')\,dt'$$

$$\hat{\phi} = K\int_{-\infty}^{t} v(t')\,dt$$

$$f_i(t) = f_c + \frac{K}{2}\sin(\phi - \hat{\phi})$$

$$\hat{\phi} = \frac{K}{2}\sin(\phi - \hat{\phi})$$

$$\frac{d\hat{\phi}}{dt} = \frac{K}{2}\sin(\phi - \hat{\phi})$$



Phase-locked Loop (PLL)

As we saw above a PLL consists of a multiplier (a mixer) a filter and a Voltage Controlled oscillator.

A VCO generates a sinusoid whose frequency deviation from the center frequency f_c is proportional to its input $v(t)$:

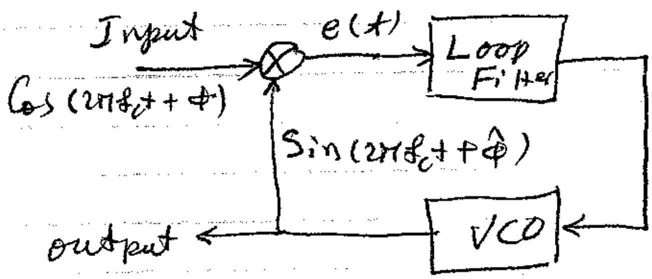
$$v(t) \longrightarrow \boxed{VCO} \longrightarrow \sin\phi_i(t) = \sin[2\pi f_c t + \hat{\phi}_t]$$

$$f_i(t) = f_c + K\,v(t)$$

Taking the integral we get

$$\phi_i(t) = \int_{-\infty}^{t} 2\pi f_i(t')\,dt' = 2\pi f_c t + K\int_{-\infty}^{t} v(t')\,dt'$$

$$= 2\pi f_c t + \hat{\phi}(t)$$

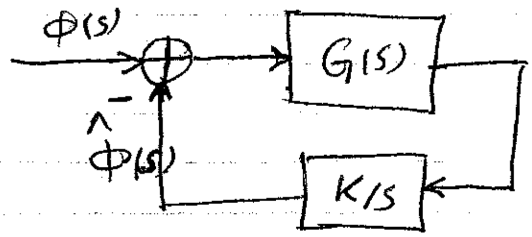


$$e(t) = \frac{1}{2}\sin(\phi - \hat{\phi}) + \frac{1}{2}\sin(2\pi f_c t + \phi + \hat{\phi})$$

If we assume that $\phi - \hat{\phi}$ (the phase error) is small, we have:

$$\sin(\hat{\phi} - \phi) \simeq \hat{\phi} - \phi$$

and we can model the PLL as:



$$\hat{\phi}(t) = K\int_{-\infty}^{t} v(t')\,dt'$$

$$\hat{\Phi}(s) = K\frac{V(s)}{s}$$

$$\frac{\hat{\Phi}(s)}{V(s)} = \frac{K}{s}$$

The closed-loop transfer function is:

$$H(s) = \frac{K\frac{G(s)}{s}}{1 + \frac{KG(s)}{s}}$$

If we assume a simple loop filter:

$$G(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

$$H(s) = \frac{1 + \tau_2 s}{1 + (\tau_2 + \frac{1}{K})s + (\frac{\tau_1}{K})s^2}$$

$$H(s) = \frac{\frac{K\tau_2}{\tau_1}s + \frac{K}{\tau_1}}{s^2 + (\frac{K\tau_2}{\tau_1} + \frac{1}{\tau_1}) + \frac{K}{\tau_1}}$$

Putting denominator in a canonic form

$$D(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

We have: $\omega_n = \sqrt{\frac{K}{\tau_1}}$ and $\xi = \omega_n(\tau_2 + \frac{1}{K})/2$.

Then:

$$H(s) = \frac{(2\xi\omega_n - \frac{\omega_n^2}{K})s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

ω_n is called natural frequency and ξ is the loop damping factor.

The (one-sided) noise-equivalent bandwidth is:

$$B_{eq} = \frac{\tau_2^2(1/\tau_2^2 + K/\tau_1)}{4(\tau_2 + 1/K)} = \frac{1 + (\tau_2\omega_n)^2}{8\xi/\omega_n}$$

The phase error $\Delta\phi$ of a first order PLL (i.e., $G(s) = 1$ and $H(s) = \frac{K}{s+K}$) is a random variable with density:

$$p(\Delta\phi) = \frac{\exp(\gamma_L \cos\Delta\phi)}{2\pi I_0(\gamma_L)}$$

where $\gamma_L = SNR = \frac{A_c^2}{N_0 B_{eq}}$

and $I_0(\cdot)$ is the modified Bessel function of order zero:

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\phi}\, d\phi = \sum_{k=0}^{\infty}\left(\frac{x^k}{2^k k!}\right)^2.$$

In the above analysis of PLL, we assumed that we have a "clean" copy of the signal, i.e., it does not carry any information. This require sending an unmodulated carrier, in form of a preamble or a tone. This means an overhead. If that is not the case, we have two options: 1) Using the detected symbols and remove the modulation from the delayed version of signal. This is called <u>Decision Directed</u> estimation. (In ~~general~~ fact one can use a preamble as another option).

2) Removing the modulation by squaring (or raising to any appropriate power).

There is yet another method that assumes the data is a random variable and tries to find the phase ϕ such that the likelihood averaged over all data values is maximized.

Decision-directed loops

Denote data $\{I_n\}$ $n = 0, \cdots, K-1$.

$$r_e(t) = e^{-j\phi} \sum_{n=0}^{K-1} I_n g(t-nT) + z(t)$$

$$\Lambda_L(\phi) = \mathrm{Re}\left\{\left[\frac{1}{N_0} \int_{T_0} r_e(t) s_e^*(t)\, dt\right] e^{j\phi}\right\}$$

$$= \mathrm{Re}\left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* \int_{nT}^{(n+1)T} r_e(t) g^*(t-nT)\, dt \right\}$$

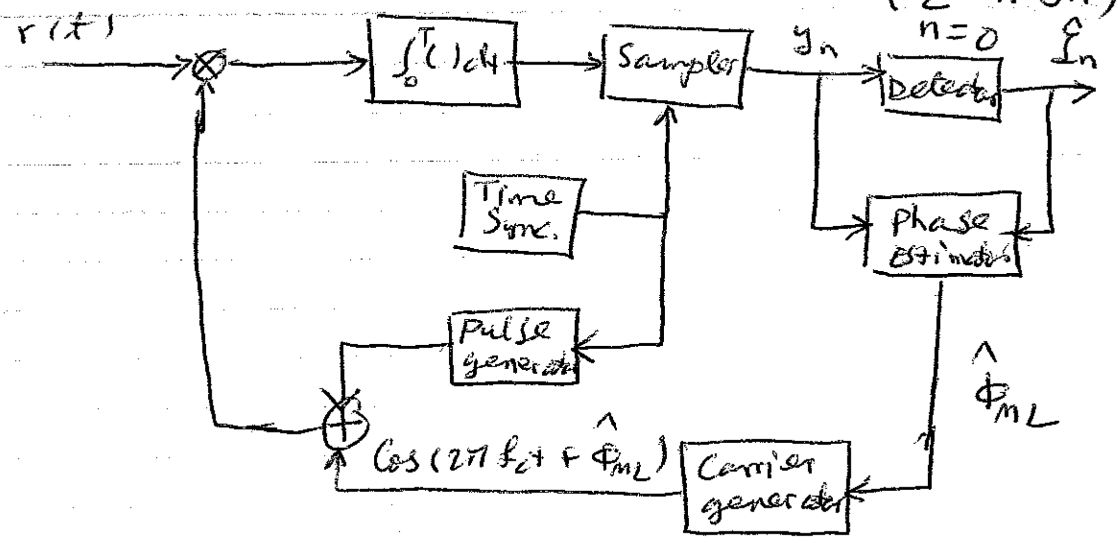
$$= \mathrm{Re}\left\{ e^{j\phi} \frac{1}{N_0} \sum_{n=0}^{K-1} I_n^* y_n \right\}$$

where

$$y_n = \int_{nT}^{(n+1)T} r_e(t) g^*(t-nT)\, dt$$

$$\Lambda_L(\phi) = \mathrm{R}\left(\frac{1}{N_0}\sum_{n=0}^{K-1} I_n^* y_n\right) \cos\phi - \mathrm{Im}\left(\frac{1}{N_0}\sum_{n=0}^{K-1} I_n^* y_n\right) \sin\phi$$

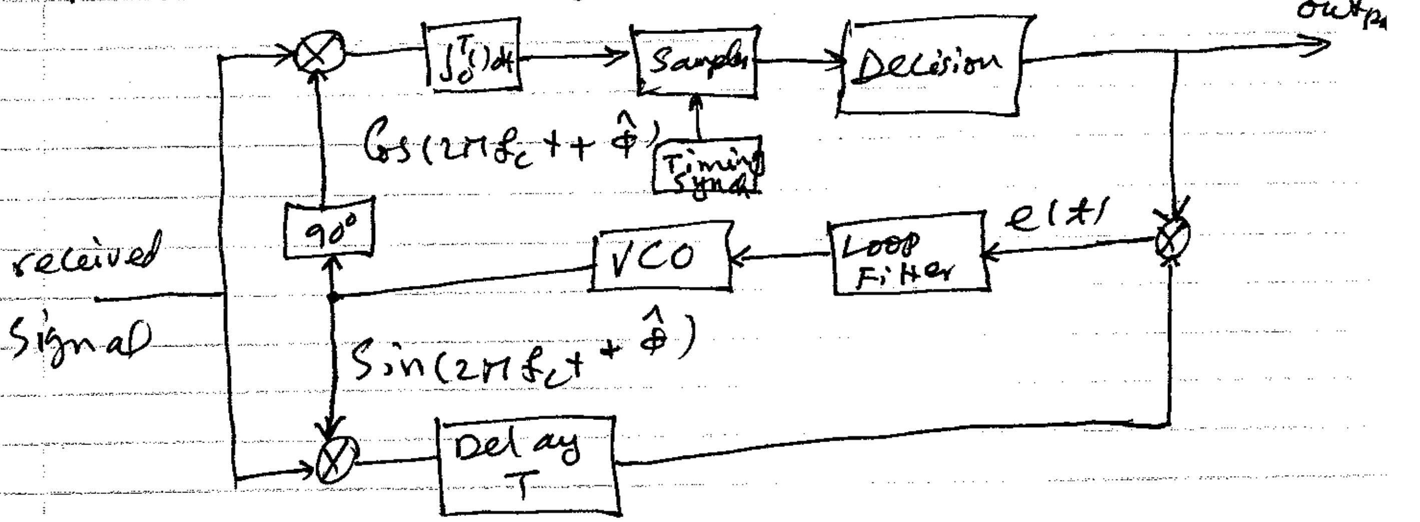
$$\frac{d}{d\phi}\Lambda_L(\phi) = 0 \Rightarrow \hat{\phi}_{ML} = -\tan^{-1} \frac{\mathrm{Im}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)}{\mathrm{Re}\left(\sum_{n=0}^{K-1} I_n^* y_n\right)}.$$



Another approach for implementing the above formula, i.e., forcing

$$\frac{d}{d\phi}\Lambda_L(\phi) = 0$$ is to use Decision

Feedback PLL (DPLL)



Non-Decision Directed Loops

Assume that we receive

$$s(t) = s_n \cos 2\pi f_c t \quad \text{where } s_n \in \{-1, +1\}$$

if the bits are equipropable, then

$$p(s_n) = \frac{1}{2}\delta(s_n - 1) + \frac{1}{2}\delta(s_n + 1)$$

and

$$\bar{\Lambda}(\phi) = \int_{-\infty}^{\infty} \Lambda(\phi) p(s_n) = \frac{1}{2}\exp\left[\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t + \phi)\,dt\right]$$

$$+ \frac{1}{2}\exp\left[-\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t + \phi)\right]dt$$

$$= \cosh\left[\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t + \phi)\,dt\right]$$

The log-likelihood function will be:

$$\bar{\Lambda}_L(\phi) = \ln \cosh\left[\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t + \phi)\,dt\right]$$

although the above function is difficult to handle, the following approximation makes life easier:

$$\ln \cosh x = \begin{cases} \frac{1}{2}x^2 & |x| \ll 1 \\ |x| & |x| \gg 1 \end{cases}$$

<u>Squaring Loop:</u>

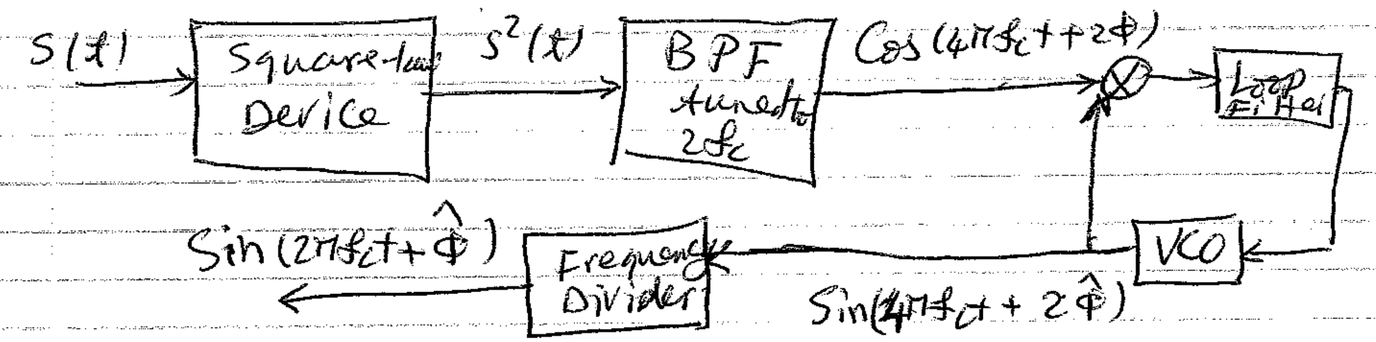
$s(t) = A(t)\cos(2\pi f_c t + \phi)$

if we raise $s(t)$ to power two, we remove the polarity of $A(t)$ (in binary case)

$$s^2(t) = A^2(t)\cos^2(2\pi f_c t + \phi)$$
$$= \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_c t + 2\phi)$$

if we pass $s^2(t)$ through a band-pass filter centered aroun $2f_c$, we get the DC)

DC cancelled and have a signal proportional to $\cos(4\pi f_c t + 2\phi)$. If we, then pass this through a PLL we get as signal $\sin(2\pi f_c t + 2\hat{\phi})$, ~~Thomsea com~~ where $\hat{\phi}$ is close to ϕ. Dividing the frequency by two, we have the required carrier.



Timing Estimation:

$$r(t) = s(t;\tau) + n(t)$$

where

$$s(t;\tau) = \sum_n I_n g(t - nT - \tau)$$

Log-likelihood function will be:

$$\Lambda_L(\tau) = C_L \int_{T_0} r(t)\, s(t;\tau)\, dt$$

$$\Lambda_L(\tau) = C_L \sum_n I_n \int_{T_0} r(t)\, g(t-nT-\tau)\, dt$$

$$= C_L \sum_n I_n\, y_n(\tau)$$

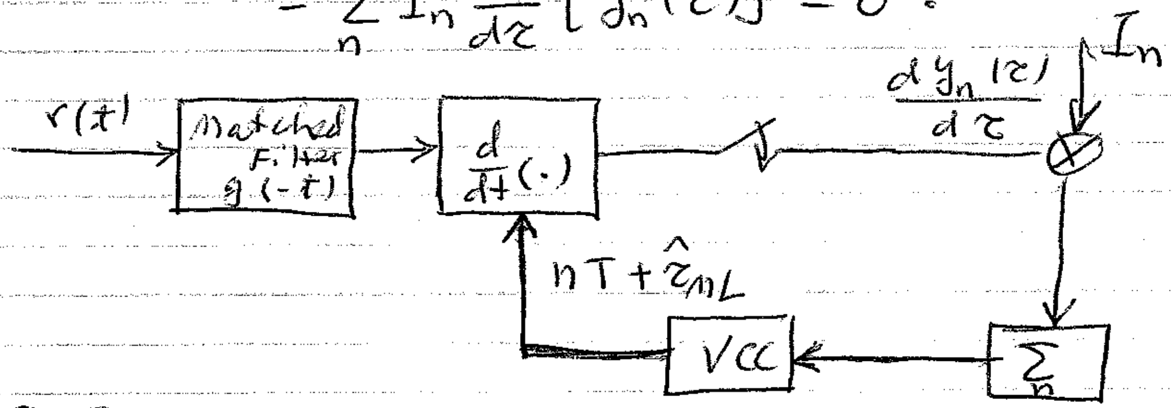
where,

$$y_n(\tau) = \int_{T_0} r(t)\, g(t-nT-\tau)\, dt$$

to get ML estimate of τ:

$$\frac{d\Lambda_L(\tau)}{d\tau} = \sum_n I_n \frac{d}{d\tau} \int_{T_0} r(t)\, g(t-nT-\tau)\, dt$$

$$= \sum_n I_n \frac{d}{d\tau}[y_n(\tau)] = 0.$$



- Non-Decision-Directed Timing Estimate

When data is unknown, we can maximize the likelihood function averaged over all possible values of data. For example, in binary case with equal probability, i.e., $P(I_n = 1) = P(I_n = -1) = \frac{1}{2}$

Then

$$\bar{\Lambda}(\tau) = \int_{-\infty}^{\infty} \Lambda(\tau)\, p(I_n)\, dI$$

$$p(I) = \frac{1}{2}\delta(I-1) + \frac{1}{2}\delta(I+1)$$

$$\bar{\Lambda}_n(\tau) = \frac{1}{2}\exp[2C y_n(\tau)] + \frac{1}{2}\exp[-2C y_n(\tau)]$$

$$= \operatorname{Cosh}[2C y_n(\tau)]$$

$$\log \Lambda_n(\tau) = \ln \operatorname{Cosh}[2C y_n(\tau)]$$

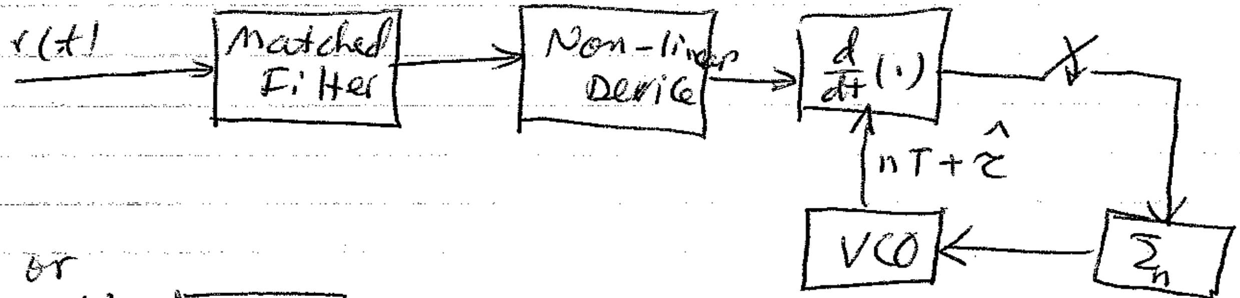
$$\bar{\Lambda}_L(\tau) = \sum_n \ln \operatorname{Cosh}[2C y_n(\tau)]$$

for $|x| \ll 1$ we have $\ln \operatorname{Cosh}(x) = \frac{x^2}{2}$

So,

$$\bar{\Lambda}_L(\tau) \simeq \frac{1}{2} C^2 \sum_n y_n^2(\tau)$$

$$\frac{d}{d\tau}\sum_n y_n^2(\tau) = 2\sum_n y_n(\tau)\frac{d y_n(\tau)}{d\tau} = 0$$



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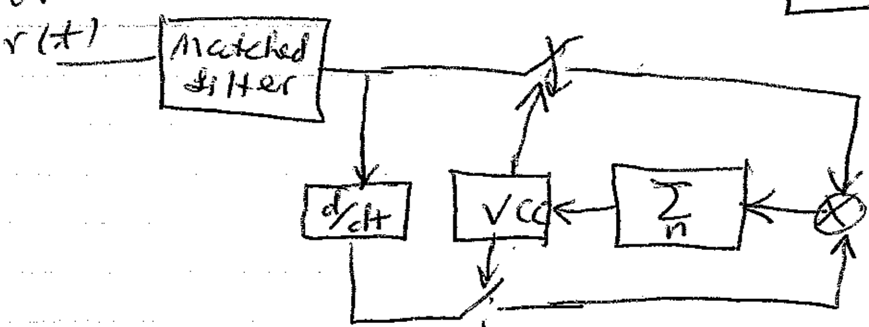
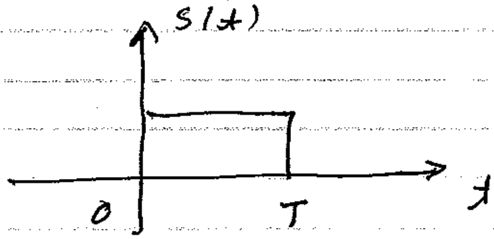
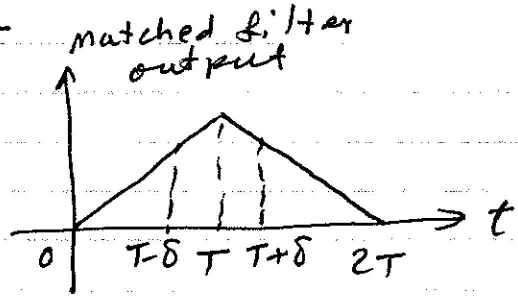


Early-late Synchronizer





We had

$$\bar{\Lambda}_L(\tau) \simeq \frac{C^2}{2} \sum_n y_n^2(\tau)$$

$$\frac{d\bar{\Lambda}_L(\tau)}{d\tau} \simeq \frac{\bar{\Lambda}_L(\tau+\delta) - \bar{\Lambda}_L(\tau-\delta)}{2\delta}$$

$$= \frac{C^2}{4\delta} \sum_n \left[y_n^2(\tau+\delta) - y_n^2(\tau-\delta) \right]$$