Signal Design for Band-limited Channels:

Take the equivalent low pass transmitted signal

$$v(t) = \sum_{n=0}^{\infty} I_n \, g_T(t-nT)$$

where $\{I_n\}$ is the information sequence and $g_T(t)$ is the pulse shaping filter (transmitter filter).

The received signal will be,

$$r(t) = \sum_{n=0}^{\infty} I_n \, h(t-nT) + z(t)$$

where $h(t)$ is the response of the channel to the pulse $g_T(t)$

$$g_T(t) \longrightarrow \boxed{c(t)} \longrightarrow h(t) = g_T(t) * c(t)$$

$$= \int_{-\infty}^{\infty} g(\tau)\, c(t-\tau)\, d\tau$$

and $z(t)$ is the AWGN.

passing $r(t)$ through the receiver filter $g_R(t)$ we get,

$$y(t) = \sum_{n=0}^{\infty} I_n \, x(t-nT) + \nu(t)$$

where $x(t) = g_T(t) * c(t) * g_R(t)$, i.e., it is the response of the receiver filter to $h(t)$.

$\nu(t)$ is the response $g_R(t)$ to $z(t)$.

Sampling at $t = kT + \tau_0$, $k = 0, 1, \ldots$, we get,

$$y(kT+\tau_0) = y_k = \sum_{n=0}^{\infty} I_n x(kT - nT + \tau_0) + \nu(kT+\tau_0).$$

or

$$y_k = \sum_{n=0}^{\infty} I_n x_{k-n} + \nu_k \qquad k = 0, 1, 2, \ldots$$

$$y = I_k x_0 + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{n-k} + \nu_k \qquad k = 0, 1, 2, \ldots$$

The first term is the desired signal and the second term is the inter-symbol-interference (ISI).

We desire to have no ISI, i.e.,

$$x_k = x(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Theorem: In order to have zero ISI, i.e.,

$$x(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

we need to have:

$$\sum_{m=-\infty}^{\infty} X\left(f+\frac{m}{T}\right) = T$$

where

$$X(f) = \mathcal{F}[x(t)]$$

Proof:

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

at $t = kT$,

$$x(kT) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f kT}\, df$$

$$= \sum_{m=-\infty}^{\infty} \int_{\frac{2m-1}{2T}}^{\frac{2m+1}{2T}} X(f)\, e^{j2\pi f kT}\, df$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{1/2T} X\left(f+\frac{m}{T}\right) e^{j2\pi f kT}\, df$$

$$= \int_{-\frac{1}{2T}}^{1/2T} \left[\sum_{m=-\infty}^{\infty} X\left(f+\frac{m}{T}\right)\right] e^{j2\pi f kT}\, df$$

$$= \int_{-\frac{1}{2T}}^{1/2T} B(f)\, e^{j2\pi f kT}\, df$$

where,

$$B(f) = \sum_{m=-\infty}^{\infty} X\left(f+\frac{m}{T}\right)$$

Note that $B(f)$ is periodic with period $\frac{1}{T}$.
So, it can be expanded as a Fourier
Series:

$$B(f) = \sum_{k=-\infty}^{\infty} b_k e^{j2\pi k f T}$$

where

$$b_k = T \int_{-\frac{1}{2T}}^{1/2T} B(f)\, e^{-j2\pi f k T}\, df$$

Comparing with the expression for $x(kT)$,

$$b_k = T\, x(-kT)$$

so if we want to have

$$x(kT) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

we need to have

$$b_k = \begin{cases} T & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Subsitituting this in $B(f) = \sum_{k=-\infty}^{\infty} b_k e^{j2\pi f k T}$

we get

$$B(f) = \sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T.$$

Assume that the channel has bandwidth of W, i.e., $C(f) = 0$, $|f| > W$

Consider the following cases:

1) $T < \frac{1}{2W}$ or $R = \frac{1}{T} > 2W$, i.e., the signalling rate exceeds $2W$.

Then $\{X(f + \frac{m}{T})\}$ consists of non-overlapping replicas of $X(f)$ so, it is not possible to have $\sum_{n=-\infty}^{\infty} X(f + \frac{m}{T}) = T$.

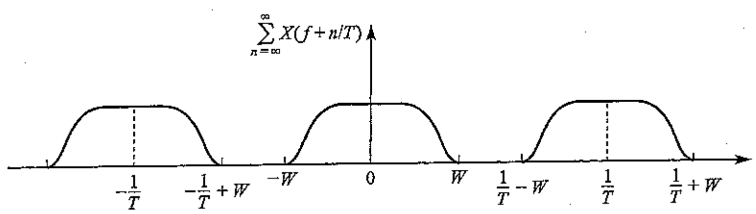
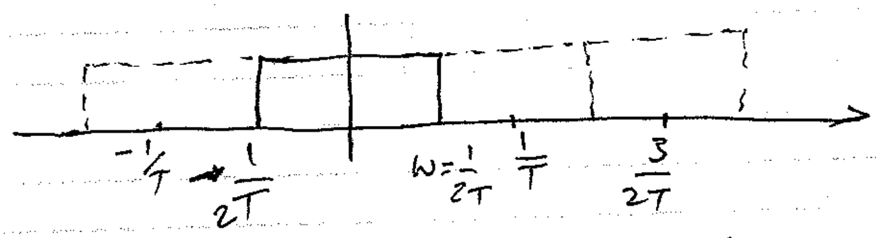


FIGURE 9.2–4
Plot of $B(f)$ for the case $T < 1/2W$.

2) $R = \frac{1}{T} = 2W$ (Nyquist rate)

In this case, we need to have a square-shaped filter (in the frequency domain):

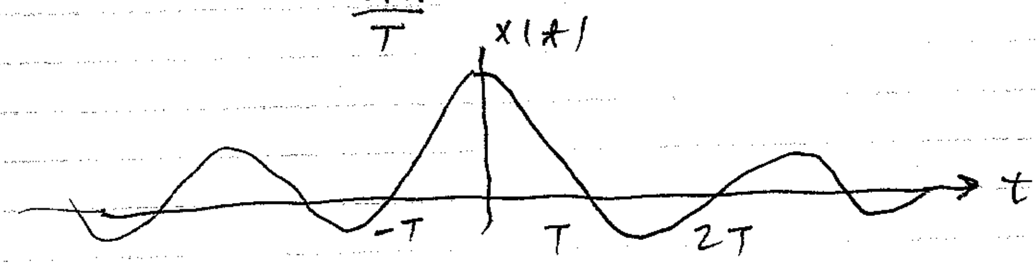


i.e.,

$$X(f) = \begin{cases} T & |f| < W \\ 0 & \text{elsewhere} \end{cases}$$

Taking inverse Fourier Transform:

$$x(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} = \mathrm{sinc}\left(\frac{\pi t}{T}\right)$$



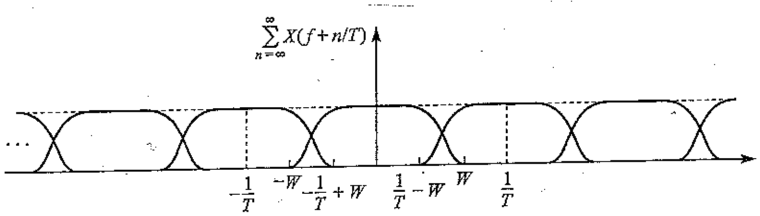
This pulse provides the minimum require bandwidth of $W = \frac{R}{2}$. However, it has the following disadvantage:

It has slow decay (as $\frac{1}{t}$) and is:

1) difficult to approximate (it has to be approximated as it is not causal).

2) Slow decay makes it very sensitive to timing error.

3) $T > \frac{1}{2W}$, then $B(f)$ consists of overlapping replicas of $X(f)$:



In this case an odd symmetry around $\frac{1}{2T}$ allows to satisfy:

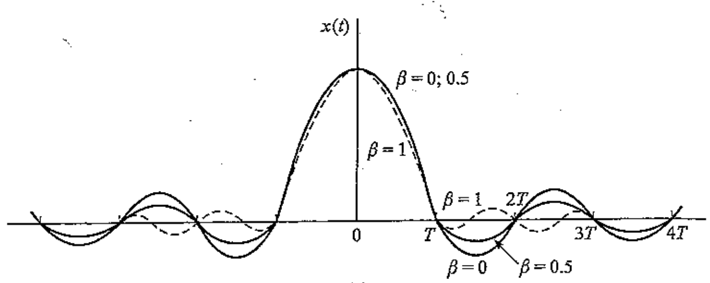
$$B(f) = \sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T$$

A pulse that satisfies this criterion is called the raise cosine pulse (or raised cosine filter).

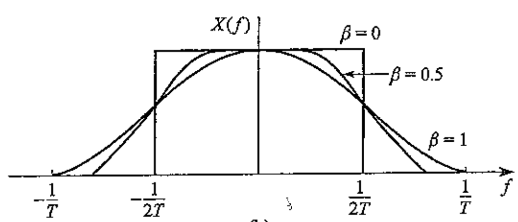
$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$

Taking the inverse Fourier Transform of $X_{rc}(f)$, we get:

$$\begin{aligned} x(t) &= \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\pi \beta t/T)}{1 - 4\beta^2 t^2/T^2} \\ &= \operatorname{sinc}(\pi t/T) \frac{\cos(\pi \beta t/T)}{1 - 4\beta^2 t^2/T^2} \end{aligned}$$



(a)



(b)

The raise cosine pulse decays as $\frac{1}{t^3}$. The rate of decay increases with β. So β (called the roll-off factor) provides a tradeoff between bandwidth efficiency and ease of implementation.

The bandwidth requirement for digital modulation:

$$W = \frac{R}{2}(1+\beta)$$

where R is the symbol rate (baud rate).

Using M-ary modulation:

$$R_b = R \log_2 M, \text{ So,}$$

$$W = \frac{R_b}{2\log_2 M}(1+\beta).$$

For bandpass (carrier) modulation, the bandwidth requirement is twice this:

$$W = \frac{R_b}{\log_2 M}(1+\beta).$$

In order to satisfy matched filter condition, it is natural to divide the raised cosine filter between the transmitter and the receiver.

The receiver filter $G_R(f)$ should be such that

$$G_T(f)\, G(f)\, G_R(f) = X_{rc}(f).$$

In the ideal case were $C(f) = 1$ for $|f| \leq W$

$$X_{rc}(f) = G_T(f)\, G_R(f)$$

or

$$G_T(f) = \sqrt{X_{rc}(f)}\, e^{-j2\pi f t_0}$$

and

$$G_R(f) = G_T^*(f).$$

Controlled ISI: Partial Response Signals

So far, we have forced the condition of zero ISI by letting $x_0 = 1$ and $x_k = 0 \quad \forall k \neq 0$

This, we saw, results in either non-realizable filter or extra bandwidth.

We may, on the other hand accept to tolerate some level of ISI to avoid frequency expansion. In particular, if we allow certain level of known (controlled) ISI by letting some $x_k \neq 0$ for som $k \neq 0$, we can avoid wasting bandwidth and, possibly, later undoing the effect of ISI.

<u>Duo-binary Signalling:</u>

Let's have

$$x(kT) = \begin{cases} 1 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$b_k = \begin{cases} T & k = 0, -1 \\ 0 & \text{otherwise} \end{cases}$$

So,

$$B(f) = T + T e^{-j2\pi fT}$$

For $T < \frac{1}{2W}$ again, it is impossible to achieve the no ISI condition. But for $T = \frac{1}{2W}$, we have

$$X(f) = \begin{cases} \frac{1}{2W}\left(1 + e^{-j\frac{\pi f}{W}}\right) & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{W} e^{-j\frac{\pi f}{2W}} \cos\frac{\pi f}{2W} & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

In time-domain:

$$x(t) = \text{Sinc}(2\pi W t) + \text{Sinc}[2\pi(Wt - \tfrac{1}{2})]$$

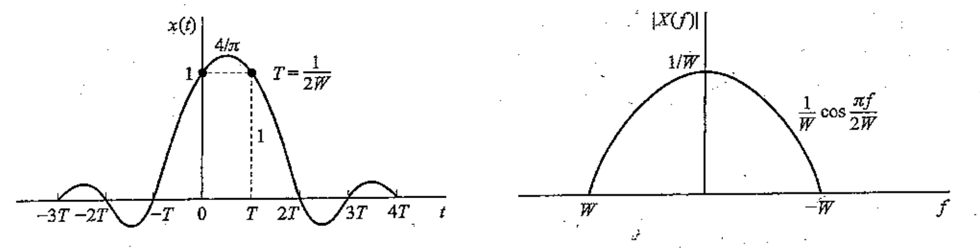


FIGURE 9.2–8
Time-domain and frequency-domain characteristics of a duobinary signal.

Letting $x_0 = x_1 = 1$ in: $(x_k = 0 \quad \forall k \neq 0 \text{ or } 1)$

$$y_k = I_k + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} + \nu_k$$

we get

$$y_k = I_k + I_{k-1} + \nu_k .$$

One problem with duo-binary pulse is that it has strong DC component. In order to rectify this problem, we can use the modified duo-binary pulse:

$$x\left(\frac{k}{2W}\right) = x(kT) = \begin{cases} 1 & k = -1 \\ -1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

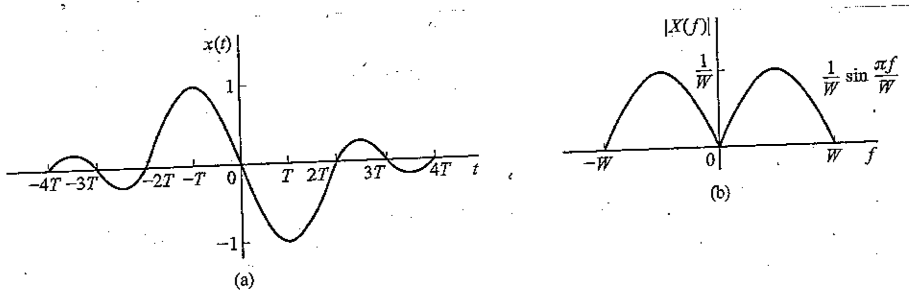
Then

$$x(t) = \text{Sinc}\,\frac{\pi(t+T)}{T} - \text{Sinc}\,\frac{\pi(t-T)}{T}$$

The spectrum is

$$X(f) = \begin{cases} \frac{1}{2W}\left(e^{j\frac{\pi f}{W}} - e^{-j\pi f/W}\right) & |f| \le W \\ 0 & |f| > W \end{cases}$$

$$= \begin{cases} \frac{j}{W} \sin \frac{\pi f}{W} & |f| \le W \\ 0 & |f| > W \end{cases}$$



In general:

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \mathrm{Sin}\left[2\pi W\left(t - \frac{n}{2W}\right)\right]$$

and

$$X(f) = \begin{cases} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\frac{n\pi f}{W}} & |f| \le W \\ 0 & |f| > W \end{cases}$$

Detection of partial response signals:

Symbol-by-Symbol detection

Take the example of duo-binary signal:

$$y_k = I_k + I_{k-1} + \nu_k$$
$$= B_k + \nu_k$$

where B_k takes values $+2, 0, -2$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

We can detect B_k and from it detect I_k as

$$I_k = B_k - I_{k-1}.$$

The problem with this method is error propagation, i.e., if one error occurs at time $t = kT$ it will make all future decisions erroneous (until another error makes the situation better).

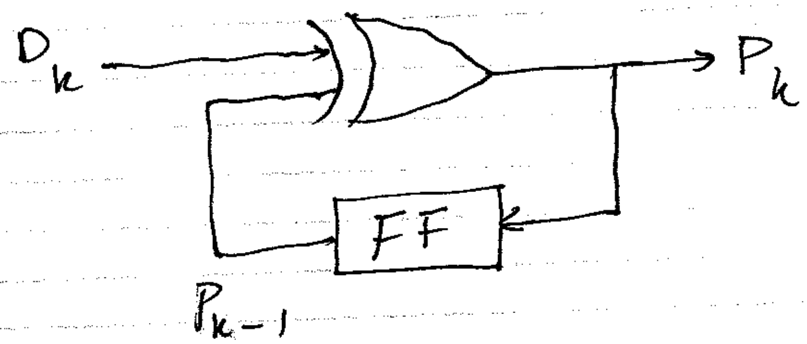
To avoid this, we do precoding.

Assume that the data to be transmitted is $\{D_k\}$.

We let

$$P_k = D_k \oplus P_{k-1}$$

i.e., at time k, we transmit on if the current input is different from the previously sent bit.



$\{D_k\}$ can be recovered from $\{P_k\}$ by

$$D_k = P_k \oplus P_{k-1}.$$

We then send $I_k = \begin{cases} 1 & \text{if } P_k = 1 \\ -1 & \text{if } P_k = 0 \end{cases}$

i.e., $I_k = (2P_k - 1)$

So,

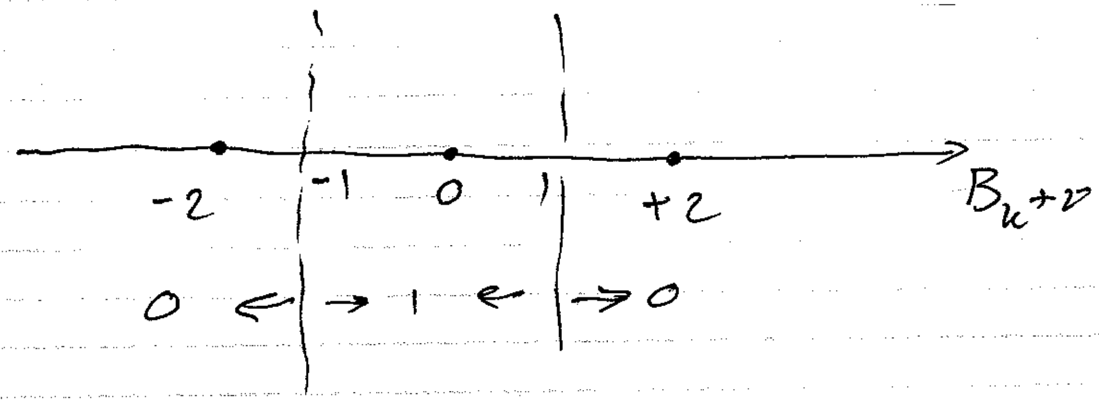
$$B_k = I_k + I_{k-1} = (2P_k - 1) + (2P_{k-1} - 1)$$
$$= 2(P_k + P_{k-1} - 1)$$

or $\boxed{P_k + P_{k-1} = \frac{1}{2}B_k + 1}$

But since $D_k = P_k \oplus P_{k-1}$, we have

$$D_k = \frac{1}{2} B_k + 1 \quad (\text{mod. } 2)$$

So, if $B_k = 2$ or $B_k = -2$ then $D_k = 0$
and if $B_k = 0$, $P_k = 1$



$$D_k = \begin{cases} 1 & |y_k| < 1 \\ 0 & |y_k| \geq 1 \end{cases}$$

where $y_k = B_k + v_k$

Example:

Data sequence	D_k:	1 1 1 0 1 0 0 1 0 0 0 1 1 0 1
precoded "	P_k:	0 1 0 1 1 0 0 0 1 1 1 1 0 1 1 0
Transmitted "	I_k:	-1 1 -1 1 1 -1 -1 -1 1 1 1 1 -1 1 1 -1
Received "	B_k:	0 0 0 2 0 -2 -2 0 2 2 2 0 0 2 0
Decoded "	$\hat{D}_k$:	1 1 1 0 1 0 0 1 0 0 0 1 1 0 1

The probability of error can be bounded as:

$$P_M < 2\left(1-\frac{1}{M^2}\right) Q\left(\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6(\log_2 M)E_b}{(M^2-1)N_0}}\right)$$

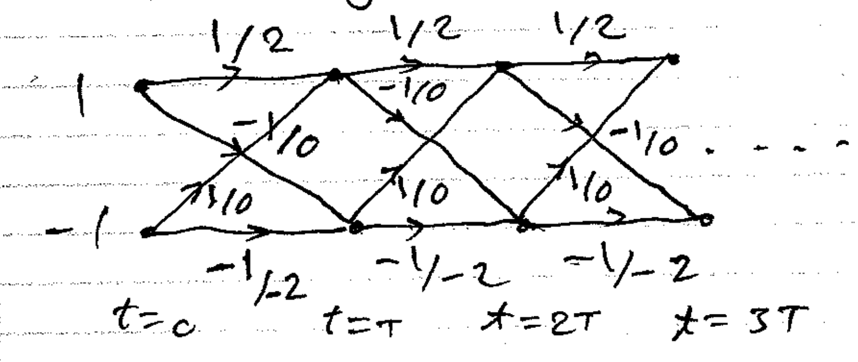
for symbol-by-symbol detection of the partial response signals.

But the probability of error for M-ary PAM is (from Chapter 4: or ELEC6831)

$$P_M \simeq \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)E_b}{(M^2-1)N_0}}\right)$$

So, the performance of partial-response signaling (with symbol-by-symbol detection) is $\left(\frac{4}{\pi}\right)^2$ worse (i.e., 2.1 dB worse).

This deficiency can be remedied using Maximum-likelihood ~~Symbol~~ sequence detection, e.g., using Viterbi Algorithm.



Optimum (Maximum-Likelihood) Receiver for Band-limited Channels

The equivalent low pass, received signal is:

$$r(t) = \sum_k I_k h(t-kT) + z(t)$$

where $\{I_k\}$ is the information sequence, $h(t)$ is the response of the channel to transmitter pulse $g_T(t)$, i.e.,

$$h(t) = g_T(t) * c(t)$$

Let's expand $r(t)$ is terms of orthonormal set $\{f_k(t)\}$ as:

$$r(t) = \lim_{N \to} \sum_{k=1}^{N} r_k f_k(t)$$

It is easy to show that

$$r_k = \sum_n I_n h_{k,n} + z_k \qquad k = 1, 2, \text{---}$$

where $h_{k,n}$ is the projection of $h(t-nT)$ onto $f_k(t)$ and z_k is the projection of $z(t)$ onto $f_k(t)$. $\{z_k\}$ is zero-mean Gaussian with covariance

$$\frac{1}{2}E[z_k^* z_m] = N_0 \delta_{k,m}$$

Let

$\underline{r}_N = [r_1, \cdots r_N]$ and $I_p = [I_1, I_2, \cdots I_p]$

$p \leq N$. Then, we have:

$$p(\underline{r}_N | \underline{I}_p) = \frac{1}{(2\pi N_0)^N} \exp\left[-\frac{1}{2N_0}\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{k,n}\right|^2\right]$$

To do ML detection, we need to find the $\underline{I}_p = (I_1, \cdots I_p)$ that maximizes $p(\underline{r}_N | \underline{I}_p)$ or equivalently maximizes:

$$-\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{k,n}\right|^2$$

as $N \to \infty$ the metric tends to Proportional Metric (PM)

$$PM(\underline{I}_p) = -\int_{-\infty}^{\infty}\left|r(t) - \sum_n I_n h(t-nT)\right|^2 dt$$

$$= \int_{-\infty}^{\infty}|r(t)|^2 dt + 2\,\mathrm{Re}\sum_n I_n^* \int_{-\infty}^{\infty} r(t) h^*(t-nT)\,dt$$

$$-\sum_n \sum_m I_n^* I_m \int_{-\infty}^{\infty} h^*(t-nT) h(t-mT)\,dt$$

Note that $|r(t)|^2$ is common to all metrics and can be discarded. So, we ~~to~~ need to maximize the Correlation Metric (CM)

$$CM(\underline{I}_p) = 2\,\mathrm{Re}\left(\sum_n I_n^* y_n\right) - \sum_n \sum_m I_n^* I_m x_{n-m}$$

where,

$$y_n = y(nT) = \int_{-\infty}^{\infty} r(t) h^*(t-nT)\,dt \qquad (A)$$

and

$$x_n = x(nT) = \int_{-\infty}^{\infty} h^*(t) h(t+nT)\,dt$$

Substituting $r(t) = \sum_n I_n h(t-nT) + z(t)$ into (A), we get:

$$y_k = \sum_n I_n x_{k-n} + \nu_k$$

where

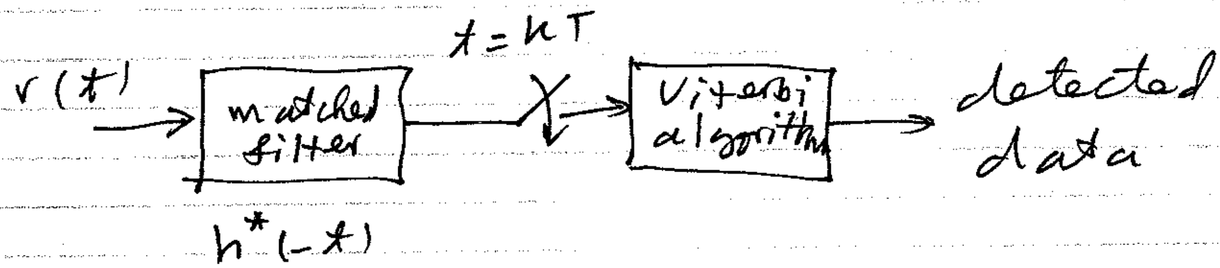
$$\nu_k = \int_{-\infty}^{\infty} z(t) h^*(t-kT)\,dt$$

We can write $CM(\underline{I}_p)$ as:

$$CM_n(\underline{I}_n) = CM_{n-1}(\underline{I}_{n-1}) + \mathrm{Re}\left[I_n^*\left(2y_n - x_0 I_n - 2\sum_{m=1}^{L} x_m I_{n-m}\right)\right]$$

where we have assumed that $x_n = 0$ for $|n| > L$.

So, it can be computed recursively using Viterbi Algorithm.

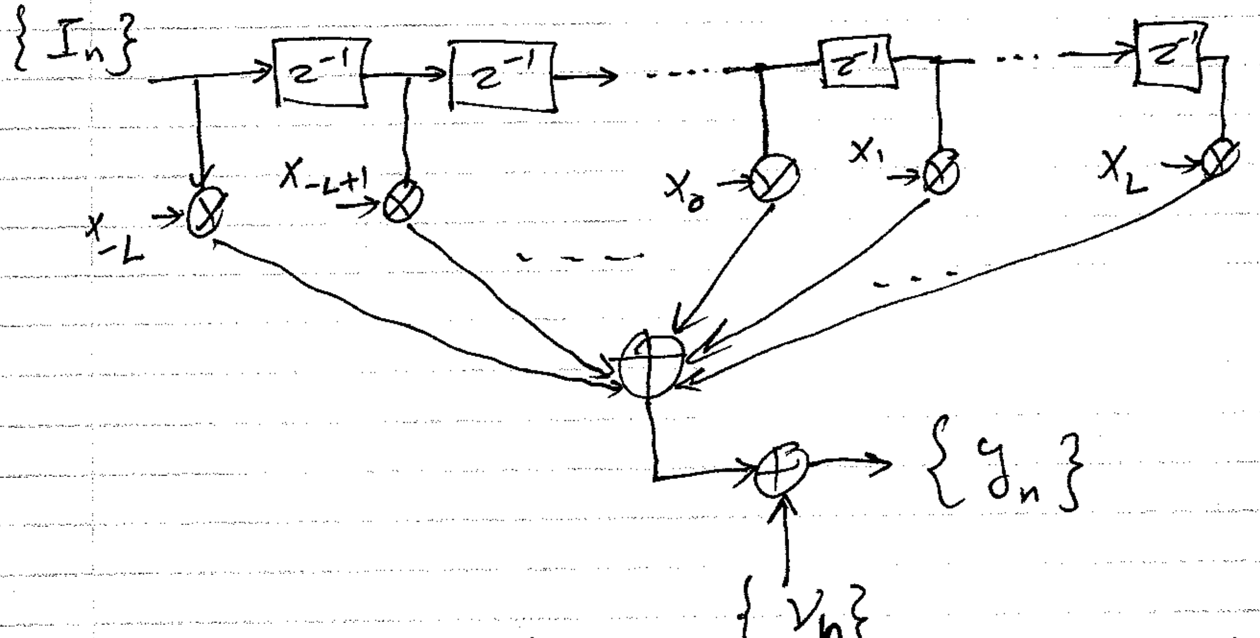


This is Maximum-Likelihood Sequence Detector (MLSD) or MLSE.

The formula

$$y_k = \sum_n I_n x_{k-n} + \nu_k$$

can be modelled as



where as before, we have assumed $x_n = 0$ for $|n| \geq L$.

$\{x_n\}$ represents a discrete-time filter consisting of the sampled version of the filter representing $g_T(t)$, $c(t)$ and the receiver filter $g_R(t) = h^*(-t)$ where $h(t) = g_T(t) * c(t)$.

Note that while $\{z_k\}$ is white, i.e.,

$$\frac{1}{2} E[z_k^* z_m] = N_0 \delta_{k,m},$$

the filtered noise components $\{\nu_k\}$ are not white. In particular,

$$\frac{1}{2} E[\nu_k^* \nu_m] = \begin{cases} N_0 x_{m-k} & |k-m| \leq L \\ 0 & \text{otherwise} \end{cases}$$

We can use a whitening filter to decorrelate $\{\nu_k\}$. Note that

$$X(z) = \sum_{k=-L}^{L} x_k z^{-k}$$

is the two sided z-transform of $\{x_k\}$

Since $x_k = x_{-k}^*$ then $X(z) = X^*(\frac{1}{z^*})$.

This means that $2L$ roots of $X(z)$ are paired such that if ρ is a root of $X(z)$, so is $\frac{1}{\rho^*}$. So, $X(z)$ can be written as:

$$X(z) = F(z)\, F^*(z^{-1}).$$

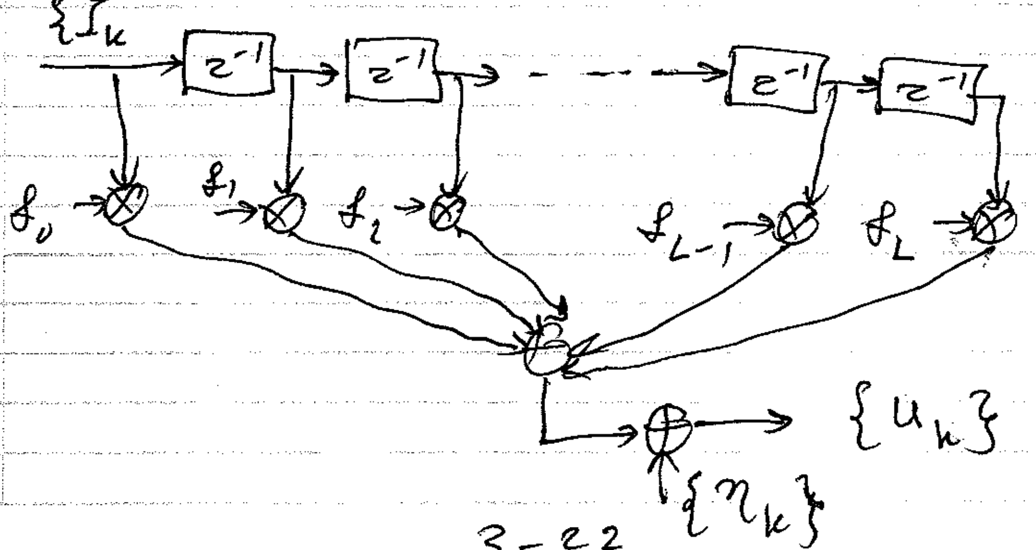
There are 2^L choices for this decomposition. Therefore, there are 2^L possibilities for the whitening filter $\frac{1}{F^*(z^{-1})}$.

We resolve this ambiguity by insisting that $F^*(z^{-1})$ have those roots that are outside the unit circle. This choice results in a stable whitening filter since $\frac{1}{F^*(z^{-1})}$ has all its poles outside the unit circle.

Passing $\{y_n\}$ through the whitening (digital) filter $\frac{1}{F^*(z^{-1})}$, we get

$$u_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

where $\{\eta_k\}$ is a white noise.



__Example:__ Let $h(t) = g(t) + a\,g(t-T)$

Then, we have

$$x_k = \begin{cases} a^* & k=-1 \\ 1+|a|^2 & k=0 \\ a & k=1 \end{cases}$$

since

$$x_k = \int_{-\infty}^{\infty} [g^*(t) + a^* g^*(t-T)][g(t-kT) + a\,g(t+(k-1)T]\,dt$$

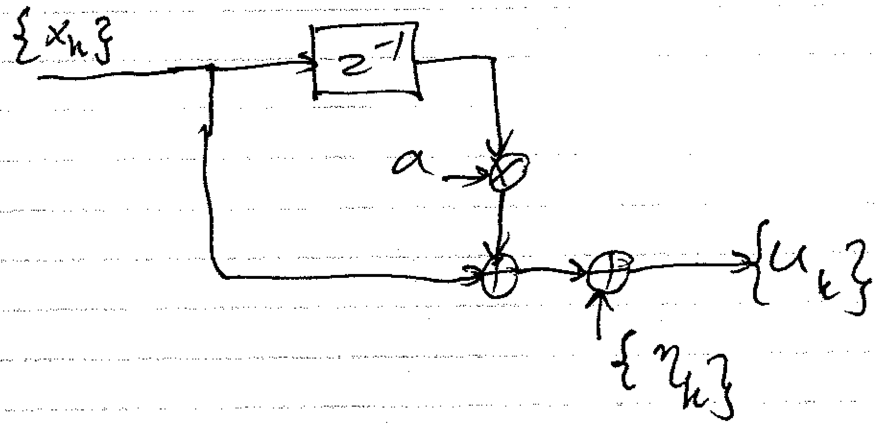
Now

$$X(z) = \sum_{k=-1}^{1} x_k z^{-k} = a^* z + (1+|a|^2) + a z^{-1}$$

$$= (a z^{-1} + 1)(a^* z + 1)$$

if $|a| < 1$, then we have $F(z) = a z^{-1} + 1$

Then $f_0 = 1$ and $f_1 = a$.



Viterbi decoding for White noise filter model:

The channel with memory of length L is a finite state machine (FSM) with states defined as

$$S_k (I_{k-1}, I_{k-2}, \cdots I_{k-L})$$

If we have M-ary signaling then $|S| = M^L$. That is we have M^L states.

— At the begining, we collect the first $L+1$ symbols (received → filtered → sampled), $u_1, u_2, \ldots, u_L, u_{L+1}$. We form the metrics:

$$\sum_{k=1}^{L+1} \ln p(u_k | I_k, I_{k-1}, \ldots, I_{k-L})$$

The M^{L+1} possible sequences are divided into M^L sets (each with M sequences) and each corresponding to one of the M^L states $(I_{L+1}, I_L, \ldots, I_2)$.

The sequence in each group, differ from each other only for I_1.

From each of the M^L groups, we select one with the highest probability:

$$PM_1(I_{L+1}) = PM_1(I_{L+1}, I_L, \dots, I_2)$$
$$= \max_{I_1} \sum_{k=1}^{L+1} \ln p(u_k \mid I_k, I_{k-1}, \dots I_{k-L})$$

The other $M-1$ sequences from each group are discarded.

- After receiving u_{L+2}, we extend the M^L surviving sequences. The probabilities of the M^{L+1} resulting sequences are calulated by adding

$$\ln p(u_{L+2} \mid I_{L+2}, I_{L+1}, \dots, I_2)$$

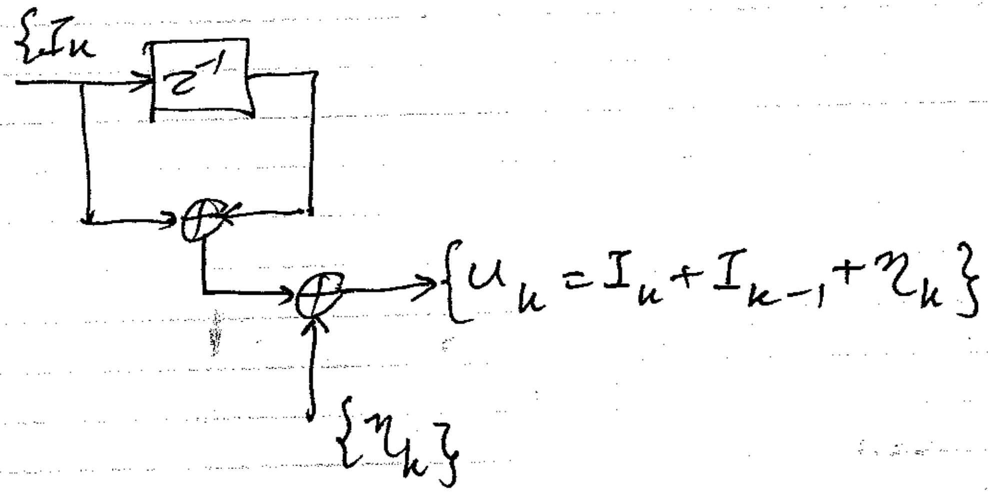
Again, we select one out of each group of M sequences. The procedure continues for $L+3, \dots$.

For k-th step

$$PM(I_{L+k}) = \max_{I_k} \left[\ln p(u_{L+k} \mid I_{L+k}, \dots, I_k) + PM_{k-1}(I_{L+k-1}) \right]$$

Example: Assume that we have a 4-level PAM with levels $\pm 1, \pm 3$ and duobinary signaling is used.

So



Assume, we have received u_1 and u_2 where,

$$u_1 = I_1 + \eta_1$$

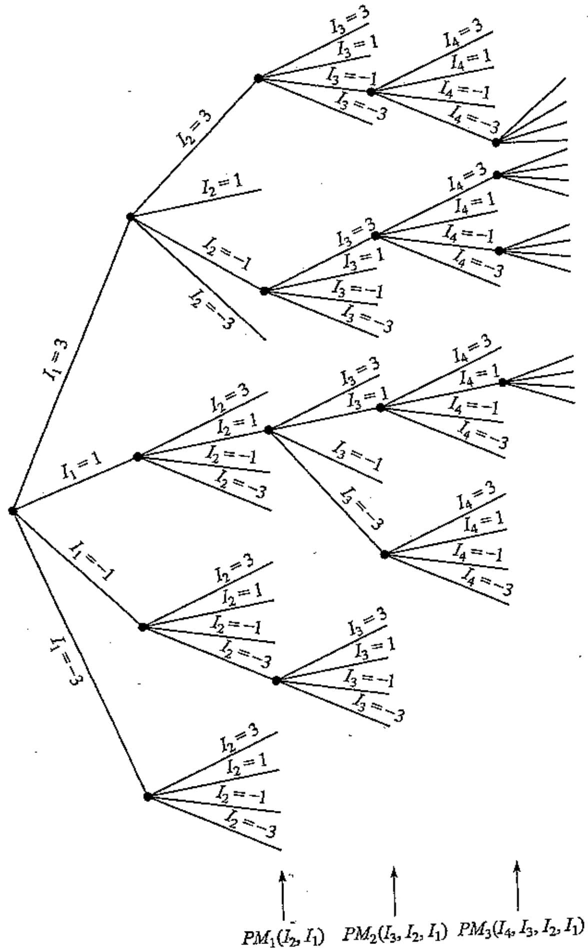
and

$$u_2 = I_2 + I_1 + \eta_2$$

we calculate 16 metrics:

$$PM(I_2, I_1) = -\sum_{k=1}^{2}\Big(u_k - \sum_{j=0}^{1} I_{k-j}\Big)^2 \qquad I_1, I_2 = \pm 1, \pm 3$$

Note that, the future values of $\{u_k\}$ do not involve I_1. So, we can discard 12 out of 16 pairs of (I_1, I_2).



For each value $i = \pm 1, \pm 3$, we find one survivor:

$$PM(I_2 = i, I_1) = \max_{I_1} \left[- \sum_{k=1}^{2} \left(u_k - \sum_{j=0}^{1} I_{k-j}\right)^2 \right]$$

After receiving u_3, we extend the four (4) metrics of the surviving paths to 16 new metrics:

$$PM(I_3, I_2, I_1) = PM(I_2, I_1) - \left(u_3 - \sum_{j=0}^{1} I_{3-j}\right)^2$$

and select one survivor for each of the values of the

$I_3 = -3$, $I_3 = -1$, $I_3 = 1$, $I_3 = 3$.

The procedure is repeated for $u_4, u_5, \ldots$

Other alternatives for implementing the receiver for channels with memory.

We saw that implementing the optimum receiver for ISI channels has a complexity that grows exponentially with the channel memory, i.e., it has a complexity M^{L+1}.

For channels with a large number of non-zero coefficients (large L) and, in particular for large constellations (large M), this complexity is prohibitive.

In order to avoid this prohibitive complexity, we can use sub-optimal detection techniques such as:

- Peak Distortion Equalizer (Zero-forcing)

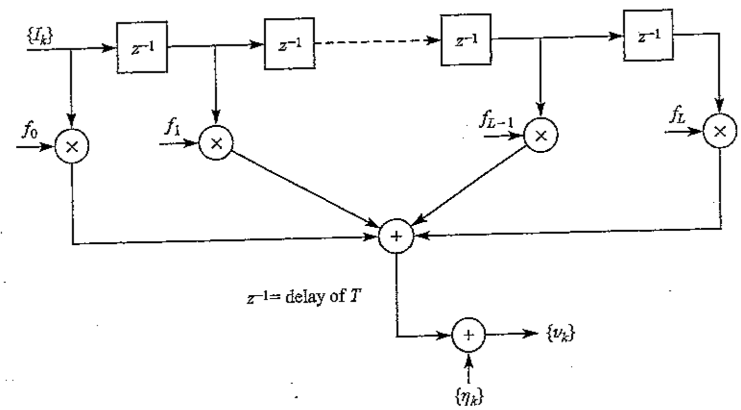
or

- Mean Square Error (MSE) equalizer.

The equation:

$$u_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

that we get for a whitened channel represents an FIR filter shown as:



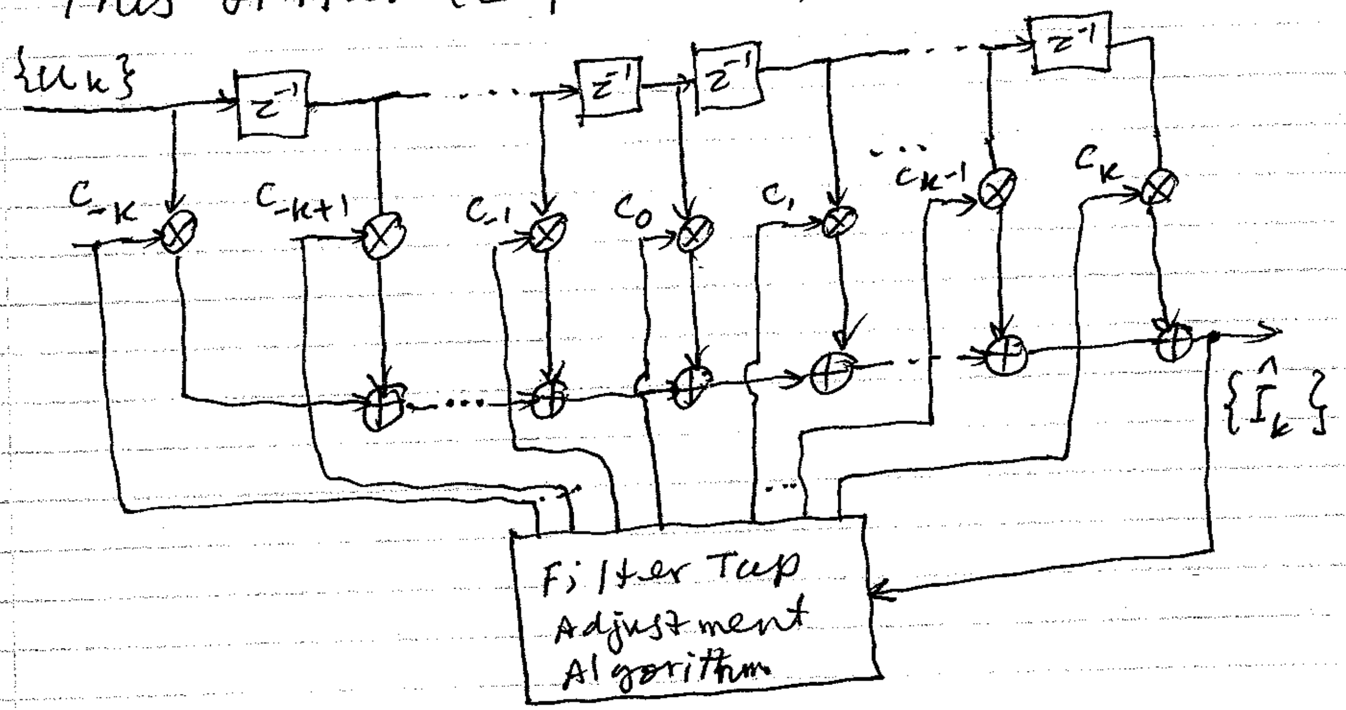
An equalizer is a filter (usually adaptive) placed in cascade with the above filter in order to compensate (equalize) the distortion that this (channel) filter causes.

Assuming that this filter look at the present output of the channel, u_k as well as K past and K future samples:

$$\hat{I}_k = \sum_{j=-K}^{K} c_j u_{k-j}$$

where $\{u_k\}$ is the set of un-equalized channel outputs, $\{c_j\}$ is the set of $2K+1$ filter coefficients and $\{\hat{I}_k\}$ is the set of equalized outputs used in final detection.

This filter (Equalizer) is shown as:



Note that the cascade of the channel filter $F(z)$ and the equalizer (filter) $C(z)$ results in a filter

$$Q(z) = F(z)C(z)$$

with coefficients:

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j}$$

That is $\{q_n\}$ is the convolution of $\{c_n\}$ and $\{f_n\}$.

Let's first assume that the equalizer has an infinite number of taps. Later, we consider the finite tap case.

With $\{q_n\}$ given as above:

$$\hat{I}_k = q_0 I_k + \sum_{n \neq k} I_n q_{n-k} + \sum_{j=-\infty}^{\infty} c_j \eta_{k-j}$$

Zero-forcing equalizer: Peak-Distortion Criterion

The first term in the above equation is the desired symbol. The second term is ISI.

The peak value of this term is:

$$D(c) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |q_n| = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \sum_{j=-\infty}^{\infty} c_j f_{n-j} \right|$$

We can choose cofficients $\{c_j\}$ such that

$$q_n = 0 \quad n \neq 0$$

$$q_n = 1 \quad n = 0$$

This is called the zero-forcing equalizer.

That is, we choose $\{c_j\}$ such that

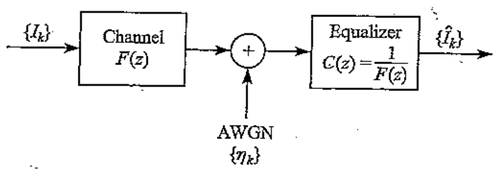
$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} = \delta_n$$

Taking z-transform of both sides, we have:

$$Q(z) = C(z)\,F(z) = 1$$

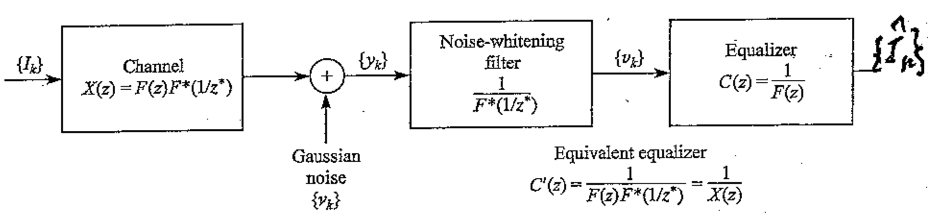
or

$$C(z) = \frac{1}{F(z)}$$



Block diagram of channel with zero-forcing equalizer.

Remember that we have used first the whitening filter $\frac{1}{F^*(z^{-1})}$ and then the zero-forcing filter $\frac{1}{F(z)}$. The combined effect is $C'(z) = \frac{1}{F(z)\,F^*(z^{-1})} = \frac{1}{X(z)}$.



The impulse response (the coefficients) of this combined filter can be found by taking the inverse z-transform of

$$C'(z) = \frac{1}{X(z)} \text{ as:}$$

$$c'_k = \frac{1}{2\pi j}\oint C'(z)\, z^{k-1}\, dz$$

$$= \frac{1}{2\pi j}\oint \frac{z^{k-1}}{X(z)}\, dz$$

<u>Performance of zero-forcing</u> equalizer:

Assume that, the received signal energy is normalized so that $q_0 = 1$ and $|I_k|^2 = 1$.

Then $\frac{1}{\sigma_n^2}$ would be the signal-to-noise ratio at the output of the equalizer.

The spectral density of the noise at the output of the equalizer is:

$$S_{\nu\nu}(\omega) = N_0 X(e^{j\omega T}) \qquad |\omega| < \frac{\pi}{T}$$

where $\quad X(e^{j\omega T}) = X(z)\big|_{z=e^{j\omega T}}$

Since $C'(z) = \frac{1}{X(z)}$, the noise power spectral density at the output of the equalizer is:

$$S_{nn}(\omega) = \frac{N_0}{X(e^{j\omega T})} \qquad |\omega| \leq \frac{\pi}{T}$$

Therefore

$$\sigma_n^2 = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\pi/T} S_{nn}(\omega)\, d\omega$$

$$= \frac{TN_0}{2\pi} \int_{-\frac{\pi}{T}}^{\pi/T} \frac{d\omega}{X(e^{j\omega})}$$

So, the SNR for infinite-tap filter is:

$$\gamma_\infty = \frac{1}{\sigma_n^2} = \left[\frac{TN_0}{\pi} \int_{-\frac{\pi}{T}}^{\pi/T} \frac{d\omega}{X(e^{j\omega})} \right]^{-1}$$

Note that

$$x_k = \int_{-\infty}^{\infty} h^*(t)\, h(t+kT)\, dt$$

from Parseval's theorem

$$x_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega kT}\, d\omega$$

where $H(\omega) = \mathcal{F}[h(t)]$

We can write the above as:

$$x_k = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right)\right|^2 \right] e^{j\omega kT} \, d\omega \qquad (A)$$

The Fourier Transform of $\{x_k\}$ is:

$$X(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} x_k \, e^{-j\omega kT}$$

where

$$x_k = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X(e^{j\omega T}) e^{j\omega T} \, d\omega \qquad (B)$$

Comparing (A) an (B), we get:

$$X(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right)\right|^2 \qquad |\omega| \leq \frac{\pi}{T}$$

So:

$$\gamma_\infty = \left[\frac{T^2 N_o}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right)\right|^2} \right]^{-1}$$

We notice that if $\sum_{n=-\infty}^{\infty} H\left(\omega + \frac{2\pi n}{T}\right)$ called the folded spectrum of $|H(\omega)|^2$ has zeros, the integral becomes infinite and the SNR tends to zero.

This means that the zero-forcing equalizer tries to remove ISI at the expense of enhancing the additive noise. The reason is that when ther is a null at a given frequency, the equalizer tries to compensate it by introducing a huge gain at that frequency. This amplifies the noise.

On the other hand if we have an ideal channel with appropriately designed signal,

$$\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 = T \qquad |\omega| \leq \frac{\pi}{T}$$

and therefore, there would be no ISI and the ~~ba~~ SNR will be:

$$\gamma_\infty = \frac{1}{N_0}$$

Finite length zero-forcing equalizer:

If we assume that the equalizer has $2K+1$ taps, the peak distortion will be:

$$D(c) = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} |q_n| = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} \left| \sum_j c_j f_{n-j} \right|$$

Since the equalizer has only $2K+1$ terms and the resulting filter $\{q_n\}$ has $2K+L$ terms, $(-K \leq n \leq K+L-1)$, it is not generally possible to make all unwanted terms equal to zero.

The peak-distortion then needs to be minimized using some standard optimization technique.

In the special case where the distortion at the input of the equalizer

$$D_0 = \frac{1}{|f_0|} \sum_{n=1}^{L} |f_n| < 1,$$

the solution is to set $q_0 = 1$ and $q_n = 0$ for $1 \leq n \leq K$. This is equivalent to using

the solution for the infinite-tap case.
Note that, in general, $\{q_n\}$ values for

$K+1 \leq n \leq K+L-1$ are not equal to

zero.

<u>Mean Square-Error</u> (MSE) equalizer:

This type of linear equalizer tries to minimize the Mean-Square-Error (MSE) between the transmitted sequence $\{I_n\}$ and the sequence at the output of the output of the equalizer $\{\hat{I}_n\}$.

Let the error ε_k at time k be,

$$\varepsilon_k = I_k - \hat{I}_k$$

The minimum MSE (MMSE) algorithm tries to minimize:

$$J = E\{|\varepsilon_k|^2\} = E\{|I_k - \hat{I}_k|^2\}$$

Let's first consider the infinite-tap MMSE equalizer:

$$\hat{I}_k = \sum_{j=-\infty}^{\infty} c_j u_{k-j} .$$

Then

$$J = E\left[\left|I_k - \sum_{j=-\infty}^{\infty} c_j u_{k-j}\right|^2\right]$$

To minimize J, we let

$$\frac{\partial J}{\partial c_\ell} = 0, \qquad \ell = 0, \pm 1, \pm 2, \cdots, \pm \infty$$

to get

$$E\left[\left(I_k - \sum_{j=-\infty}^{\infty} c_j u_{k-j}\right) u_{k-\ell}^*\right] = 0$$

or

$$\sum_{j=-\infty}^{\infty} c_j E(u_{k-j} u_{k-\ell}^*) = E[I_k u_{k-\ell}] \qquad (C)$$

$$-\infty \leq \ell \leq \infty$$

use

$$u_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

to get

$$E[u_{k-j} u_{k-\ell}^*] = \sum_{n=0}^{\infty} f_n^* f_{n+\ell-j} + N_0 \delta_{\ell j} \qquad (D)$$

$$= \begin{cases} x_{\ell-j} + N_0 \delta_{\ell j} & -L \leq \ell \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[I_k U^*_{k-\ell}] = \begin{cases} f^*_{-\ell} & -L \leq \ell \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (E)$$

Taking z-transform of both sides of (C) we get

$$C(z)\left[F(z)F^*(z^{-1}) + N_0\right] = F^*(z^{-1})$$

where we have substituted (D) and (E). So, the equalizer will have the transfer function:

$$C(z) = \frac{F^*(z^{-1})}{F(z)F^*(z^{-1}) + N_0}$$

Combinining this equalizer with the whitening filter, we get:

$$C'(z) = \frac{1}{F^*(z^{-1})F(z) + N_0} = \frac{1}{X(z) + N_0}.$$

We observe that unlike the zero-forcing equalizer, the MMSE equalizer takes into account the channel noise.

The performance of the MMSE equalizer:

$$J_{min} = E[\varepsilon_k I_k^*]$$

$$= E[|I_k|^2] - \sum_{j=-\infty}^{\infty} c_j E[U_{k-j} I_k^*]$$

$$= 1 - \sum_{j=-\infty}^{\infty} c_j \cdot f_{-j}$$

The term $\sum_{j=-\infty}^{\infty} c_j \cdot f_{-j}$ is the convolution of $\{c_j\}$ and $\{f_j\}$ at a shift zero.

If we denote the convolution of $\{c_j\}$ and $\{f_j\}$ by $\{b_j\}$. Then $\sum_{j=-\infty}^{\infty} c_j f_{-j} = b_0$

$$B(z) = \mathcal{Z}[\{c_j\} * \{f_j\}] = C(z) F(z)$$

$$= \frac{F(z) F^*(z^{-1})}{F(z) F^*(z^{-1}) + N_0} = \frac{X(z)}{X(z) + N_0}$$

The term b_0 is:

$$b_0 = \frac{1}{2\pi j} \oint \frac{B(z)}{z} dz = \frac{1}{2\pi j} \oint \frac{X(z)}{z[X(z) + N_0]} dz$$

Letting $z = e^{j\omega T}$, the above contour integral will be transformed to:

$$b_0 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega$$

$$J_{min} = 1 - \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{X(e^{j\omega T}) + N_0} d\omega$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{\frac{1}{T} \sum_{n=-\infty}^{\infty} |H(\omega + \frac{2\pi n}{T})|^2 + N_0} d\omega$$

In the absence of ISI, we have $X(e^{j\omega T}) = 1$ and

$$J_{min} = \frac{N_0}{N_0 + 1}$$

and

$$\gamma_\infty = \frac{1 - J_{min}}{J_{min}}$$

Finite length MMSE equalizer:

$$\hat{I}_k = \sum_{j=-K}^{K} c_j u_{k-j}$$

$$J(K) = E\left[|I_k - \hat{I}_k|^2\right] = E\left[\left|I_k - \sum_{j=-K}^{K} c_j u_{k-j}\right|^2\right]$$

minimizing $J(K)$ with respect to $\{c_j\}$, we get

$$(F) \quad \sum_{j=-K}^{K} c_j \Gamma_{\ell j} = \xi_\ell \qquad \ell = -K, \ldots, -1, 0, 1, \ldots, K.$$

where

$$\Gamma_{\ell j} = \begin{cases} x_{\ell-j} + N_0 \delta_{\ell j} & |\ell - j| \leq L \\ 0 & \text{otherwise} \end{cases}$$

and

$$\xi_\ell = \begin{cases} f^*_{-\ell} & -L \leq \ell \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

We can write (F) as:

$$\underline{\Gamma}\, \underline{C} = \underline{\xi}$$

where $\underline{c}$ is a column vector containing $2K+1$ filter taps, $\underline{\Gamma}$ is the $(2K+1)$ by $(2K+1)$ Hermitian matrix containing Covariances of $\{u_k\}$

that is, Γ_{kj} and $\underline{\xi}$ is $(2K+1)$-dimensional column vector having elements $\xi_j = f^*_{-j}$.
The optimal value of the coefficients can be found using

$$C_{opt} = \Gamma^{-1}\underline{\xi}$$

The minimum value of $J(K)$ is:

$$J_{min}(K) = 1 - \sum_{j=-K}^{0} c_j f_{-j}$$

$$= 1 - \xi^H \Gamma^{-1} \underline{\xi}$$

where ξ^H is the Hermitian transpose of $\underline{\xi}$.

<u>Example</u>: Consider

$$F(z) = f_0 + f_1 z^{-1}$$

where f_0 and f_1 are normalized such that $|f_0|^2 + |f_1|^2 = 1$

Then $X(z) = F(z)F^*(z^{-1}) = f_0 f_1^* z + 1 + f_0^* f_1 z^{-1}$

$$X(e^{j\omega T}) = f_0 f_1^* e^{j\omega T} + 1 + f_0^* f_1 e^{-j\omega T}$$

$$= 1 + 2|f_0||f_1| \cos(\omega T + \theta)$$

where θ is the angle of $f_0 f_1^*$

A linear MMSE equalizer will have

$$J_{min} = \frac{N_0}{\sqrt{N_0^2 + 2N_0(|f_0|^2 + |f_1|^2) + (|f_0|^2 - |f_1|^2)^2}}$$

$$= \frac{N_0}{\sqrt{N_0^2 + 2N_0 + (|f_0|^2 - |f_1|^2)^2}}$$

where we have used

$$J_{min} = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\pi/T} \frac{N_0}{X(e^{j\omega}) + N_0} d\omega$$

Assuming $f_0 = f_1 = \frac{1}{\sqrt{2}}$, we have:

$$J_{min} = \frac{N_0}{\sqrt{N_0^2 + 2N_0}} = \frac{1}{\sqrt{1 + \frac{2}{N_0}}}$$

and

$$\gamma_\infty = \sqrt{1 + \frac{2}{N_0}} - 1 \simeq \left(\frac{2}{N_0}\right)^{1/2} \quad N_0 \ll 1.$$

Comparing this with $\frac{1}{N_0}$ for the case of no ISI, we see that this channel causes lots of degradation.
For example for $N_0 = 10^{-5}$, we have SNR of 50 dB when there is no ISI while we have $\gamma_\infty = 26.5$ dB when there is ISI. That is 23.5 dB performance loss.

Example:

$$f_k = \sqrt{1-a^2}\, a^k \qquad k = 0, 1, \ldots$$

where $a<1$. Fourier Transform of this sequence is:

$$X(e^{j\omega T}) = \frac{1-a^2}{1+a^2-2a\cos\omega T}$$

The SNR can be found to be:

$$\gamma_\infty = \left[\sqrt{1+2N_0\frac{1+a^2}{1-a^2}+N_0^2}-1\right]^{-1} \simeq \frac{1-a^2}{(1+a^2)N_0} \qquad N_0 \ll 1$$

The loss due to the presence of interference (compared to $\frac{1}{N_0}$) is $-10\log\left(\frac{1-a^2}{1+a^2}\right)$.