Adaptive Equalizers

In previous lectures, we discussed different equalization techniques.

Note that: those techniques relied on the knowledge of the channel. In the landline case where the channel properties are fixed, this should not cause a problem. However, in the case of wireless communications in particular, mobile communications, the channel characteristics changes over time, we need to use adaptive schemes for compensating for the channel characteristics ~~as~~ changes.

Adaptive zero-forcing algorithm:

For zero-forcing equalizer, we force the cross correlation between the error sequence $\varepsilon_k = I_k - \hat{I}_k$ and the desired sequence $\{I_k\}$ to be zero: for $0 \le |j| < K$

$$E[\varepsilon_k I_{k-j}^*] = E[(I_k - \hat{I}_k) I_{k-j}^*] \qquad j = -K, \dots, K$$

$$= E[I_k I_{k-j}^*] - E[\hat{I}_k I_{k-j}^*]$$

We assume that the information symbols are un-correlated, i.e., $E[I_k I_{k-j}^*] = \delta_j$

Using $\hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j} = q_0 I_k + \sum_{k \neq n} I_k q_{n-k} + \sum_{j=-K}^{K} c_j \eta_{n-j}$

we get

$$E[\varepsilon_k I_{k-j}^*] = \delta_j - q_j$$

So, we get $E[\varepsilon_k \hat{I}_{k-j}] = 0$ by letting

$$q_0 = 1$$

and

$$q_j = 0 \qquad 1 \leq |j| \leq K$$

When the channel is unknown, $E[\varepsilon_k I_{k-j}^*]$ is also unknown. We need to send a known sequence $\{I_n\}$ to train the equalizer.

Then, the coefficients, found through training, can be modified recursively, using, for example:

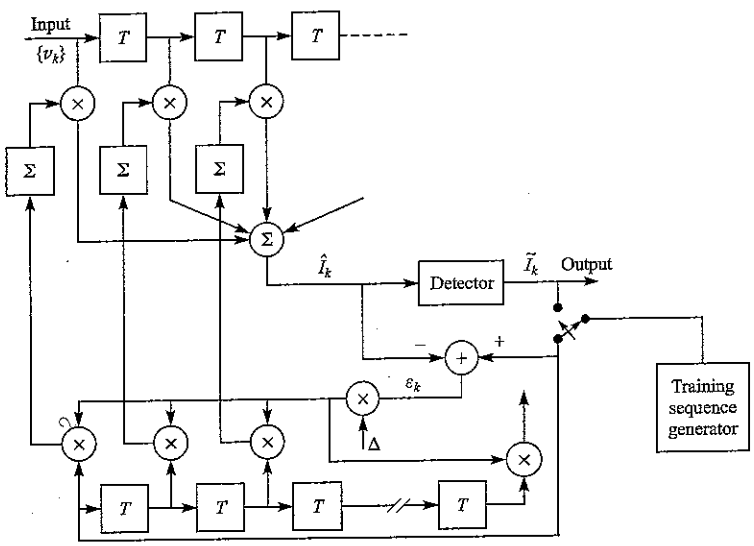
$$c_j^{(k+1)} = c_j^{(k)} + \Delta \varepsilon_k I_{k-j}^* \qquad j = -K, \cdots, 0, \cdots K$$

where Δ is a scale factor that provides a tradeoff between the speed of convergence and accuracy of the corrections.

Figure below shows an implementation of the above scheme, where, at the beginning the switch is connected to the stored training sequence and later is connected to the output, $\tilde{I}_k$. So, after the training phase

$$c_j^{(k+1)} = c_j^{(k)} + \Delta \tilde{\varepsilon}_k \tilde{I}^*_{k-j}$$

where $\tilde{\varepsilon}_k = \tilde{I}_k - \hat{I}_k$



Least Mean Square (LMS) algorithm

The minimum MSE algorithm finds the vector of equalizer coefficients $\underline{C}$ ~~as~~ by solving

$$\Gamma C = \xi$$

where Γ is $(2K+1) \times (2K+1)$ matrix of covariances of $\{v_k\}$ and ξ is a $(2K+1)$-dimensional vector containing channel coefficients.

We can, alternatively use, the recursive formula,

$$\underline{C}_{k+1} = \underline{C}_k - \Delta \underline{G}_k$$

where

$\underline{C}_k$, $\underline{C}_{k+1}$ are the vector of coefficients at iterations k and $k+1$ and $\underline{G}_k$ is

$$\underline{G}_k = \frac{1}{2} \frac{dJ}{dC_k} = \Gamma \underline{C}_k - \xi = -E[\varepsilon_k V_k^*]$$

When channel characteristics are not known a priori, i.e, when Γ, ξ and, hence $\underline{G}_k$

are not available, we use estimate of $\underline{G}_k$:

$$\hat{C}_{k+1} = \hat{C}_k - \Delta \hat{G}_k$$

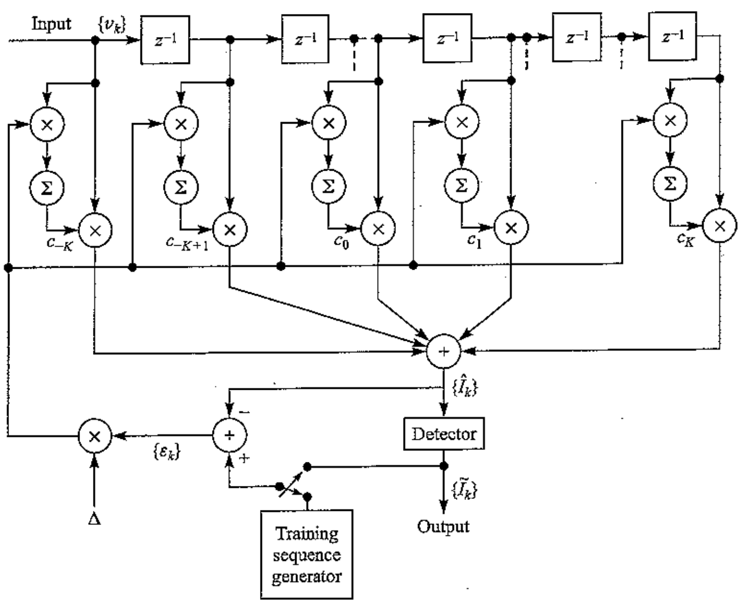
where $\hat{G}_k$ is the estimate of G_k ~~defined as~~ given by:

$$\hat{G}_k = -\varepsilon_k V_k^*$$

where $V_k = [v_{k+K}, \dots v_k, \dots v_{k-K}]^T$
is a vector of received ~~signals as~~ samples used to estimate $\hat{I}_k$.

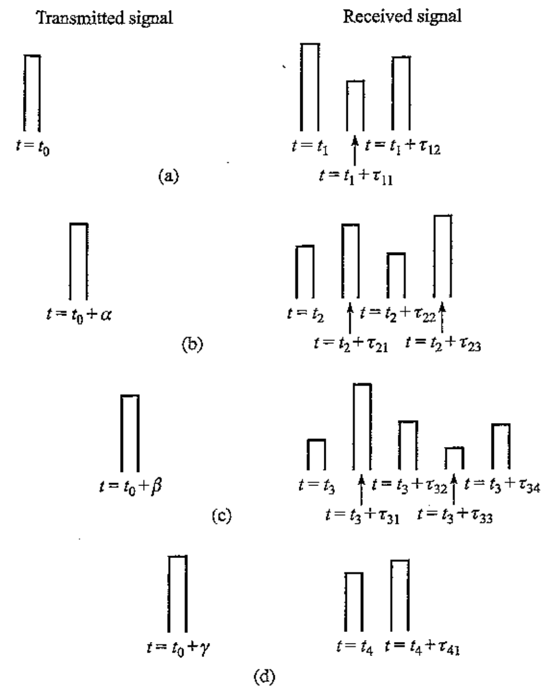
So,

$$\hat{C}_{k+1} = \hat{C}_k + \Delta \varepsilon_k V_k^*$$



Multi-Path Fading Channels

~~Unlike~~ In satellite channels, where communication is un-obstructed, i.e., it is line of sight (LOS) and, as a result, there is a single path between the transmitter and the receiver. In terrestrial links, where there are objects between the transmitter and the receiver, a signal transmitted may arrive at different times. These copies of the signal undergo different channel effects, hence, will arrive with different <u>phase</u> and <u>attenuation</u>. Furthermore, in the case of mobile wireless communications, the number of paths and the characteristics of each may change over time.



Denote the transmitted signal by $s(t)$:

$$s(t) = Re\left[s_\ell(t) e^{j2\pi f_c t}\right]$$

where $s_\ell(t)$ is the equivalent baseband signal.

The output ~~of~~ for the channel to $s(t)$ will be:

$$x(t) = \sum_n \alpha_n(t) s(t - \tau_n(t))$$

where $\alpha_n(t)$ and $\tau_n(t)$ are the attenuation and propagation delay of the n-th path at time t.

Substituting $s(t)$ in the expression for $x(t)$:

$$x(t) = Re\left(\left\{\sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} s_e[t - \tau_n(t)]\right\} e^{j2\pi f_c t}\right)$$

In the absence of noice, the received baseband signal will be:

$$r_\ell(t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} s_\ell[t - \tau_n(t)]$$

So, the channel can be characterized by a time-varying impulse response:

$$c(\tau; t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} \delta[t - \tau_n(t)]$$

To take transmission effects into consideration, one may consider transmission of an unmodulated carrier, i.e.,

$$r_e(t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)}$$

$$= \sum_n \alpha_n(t) e^{j\theta_n(t)}$$

$\alpha_n(t)$ represents signal fading.

When $c(\tau;t)$ is modelled as zero-mean Gaussian the envelope $|c(\tau;t)|$ is distributed as Rayleigh and the channel is called Rayleigh Fading Channel.

Channel Correlation Functions and Power Spectra.

The autocorrelation of $c(\tau;t)$ is:

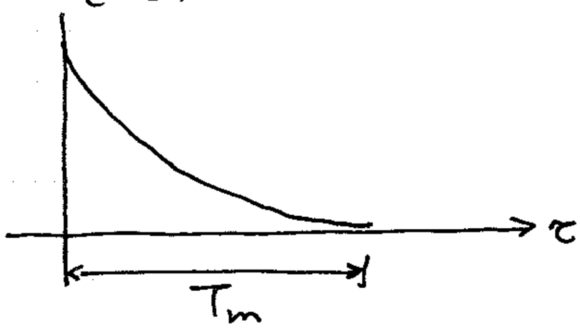
$$R_c(\tau_2,\tau_1;\Delta t) = E[c^*(\tau_1;t)c(\tau_2;t+\Delta)]$$

$$= R_c(\tau_1;\Delta t)\,\delta(\tau_2-\tau_1)$$

Where we have assumed that the attenuation and phase shift of the channel at two different phase delays τ_1 and τ_2.

Letting $\Delta t = 0$, we get $R_c(\tau;0) = R_c(\tau)$ which is the average power output of the channel for delay τ.

$R_c(\tau)$ is called the Multipath intensity profile:



The range of values of τ, where $R_c(\tau)$ is essentially non-zero (above certain small value), T_m, is called the multipath spread of the channel.

Taking the Fourier Transform of $c(\tau;t)$

$$C(f;t) = \int_{-\infty}^{\infty} c(\tau;t) e^{-j2\pi f\tau} d\tau$$

we get the time-varying transfer function of the channel $C(f;t)$.

Assuming the channel is Wide-Sense Stationary we define the autocorrelation function:

$$R_c(f_2, f_1; \Delta t) = E[C^*(f_1; t)\, C(f_2; t+\Delta t)]$$

It can be shown that:

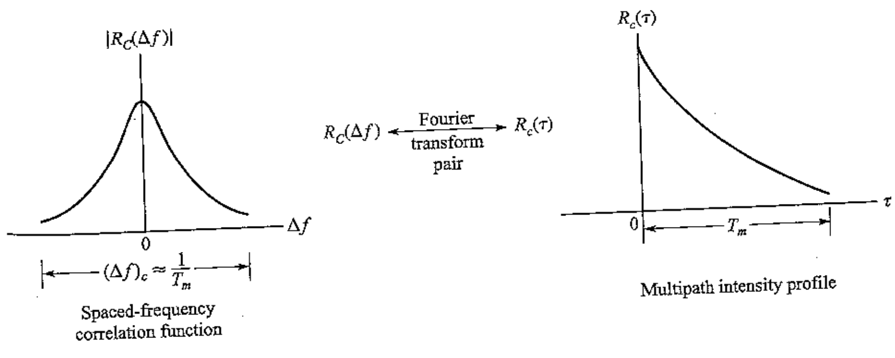
$$\begin{aligned} R_C(f_2, f_1; \Delta t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[c^*(\tau_1; t)c(\tau_2; t+\Delta t)\right] e^{j2\pi(f_1\tau_1 - f_2\tau_2)} d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_c(\tau_1; \Delta t)\delta(\tau_2 - \tau_1) e^{j2\pi(f_1\tau_1 - f_2\tau_2)} d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} R_c(\tau_1; \Delta t) e^{j2\pi(f_1 - f_2)\tau_1} d\tau_1 \\ &= \int_{-\infty}^{\infty} R_c(\tau_1; \Delta t) e^{-j2\pi \Delta f \tau_1} d\tau_1 \equiv R_C(\Delta f; \Delta t) \end{aligned}$$

where $\Delta f = f_2 - f_1$.

That is $R_c(\Delta f; \Delta t)$ is the Fourier Transform of the intensity profile.

Letting $\Delta t = 0$, we get

$$R_c(\Delta f) = R_c(\Delta f; 0) = \int_{-\infty}^{\infty} R_c(\tau) e^{-j2\pi\Delta f \tau} d\tau$$



The range of values over which $|R_c(\Delta f)|$ is essentially non-zero is called coherence bandwidth. This quantity denoted as $(\Delta f)_c$ is related to T_m as:

$$(\Delta f)_c \simeq \frac{1}{T_m}$$

The interpretation of $(\Delta f)_c$:

The ~~auto~~correlation between two ~~signals~~ realizations of the channel separated by more than $(\Delta f)_c$ is small that means that two signals apart by more than $(\Delta f)_c$ are treated differently by the channel. So, if a signal has a bandwidth greater than $(\Delta f)_c$, it see different fadings for different parts of it. In that case, the channel is said to be frequency-selective. Otherwise, the channel will be a flat fading channel. (or frequency non-selective).

Now, let's take the Fourier Transform of $R_c(\Delta f; \Delta t)$ with respect to Δt:

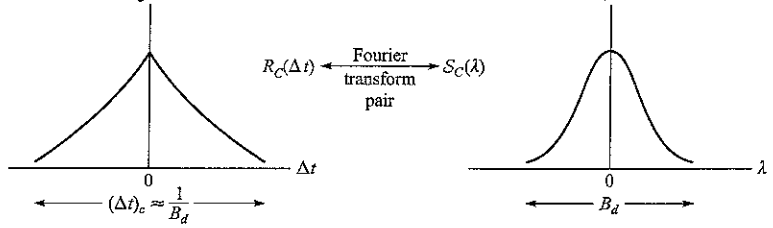
$$S_c(\Delta f; \lambda) = \int_{-\infty}^{\infty} R_c(\Delta f; \Delta t)\, e^{-j2\pi\lambda\Delta t}\, d(\Delta t)$$

Letting $\Delta f = 0$, we get

$$S_c(\lambda) = \int_{-\infty}^{\infty} R_c(0; \Delta t)\, e^{-j2\pi\lambda\Delta t}\, d(\Delta t)$$

This is called Doppler power Spectrum of the channel since $S_c(\lambda)$ gives the signal intensity as a function of Doppler frequency λ.

The range of values of λ over which $S_c(\lambda)$ is greater than some small values is called the coherence time.



Spaced-time correlation fnction

Doppler power spectrum

The coherence time is the duration of time over which the characteristics of the channel have certain level of correlation, i.e., symbols closer to each other ~~are~~ than $(\Delta t)_c$ are treated almost the same. If the rate of signal is such that the symbols are closer than $(\Delta t)_c$ we say that the channel is slow fading. Otherwise, it is fast fading.

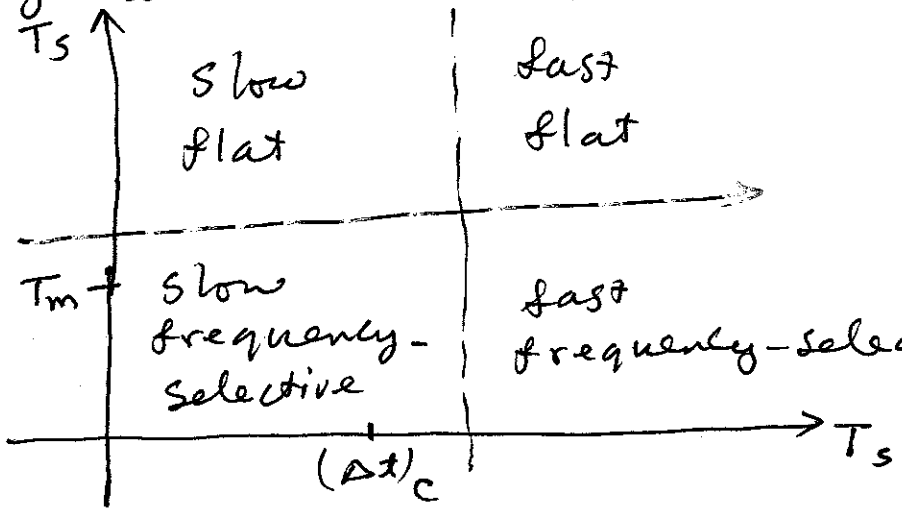
Relationship between the transmission rate (Bandwidth) and the channel model:

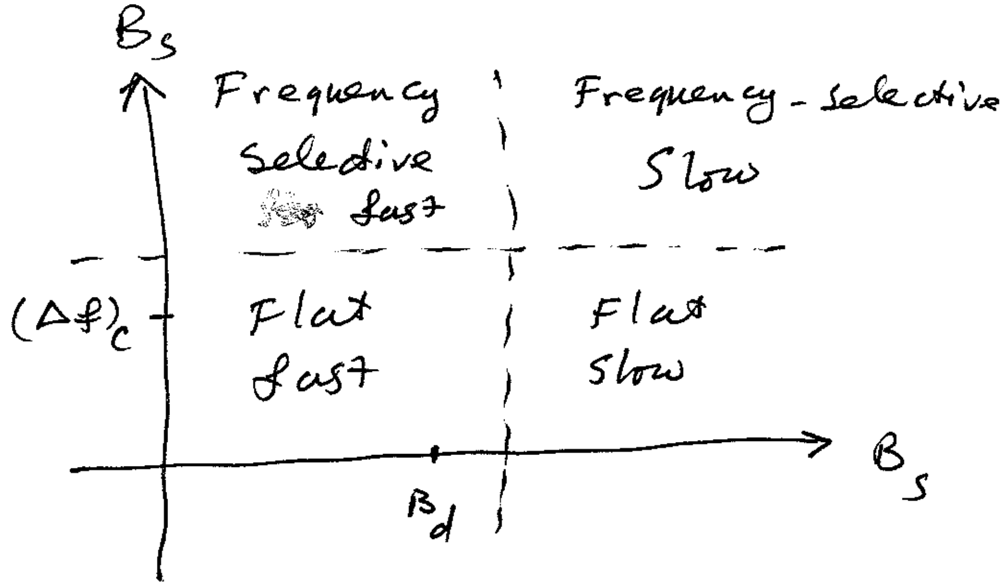
if $T_s = \frac{1}{B_c} \ll (\Delta t)_c$ or $\frac{1}{T_s} = B_c \gg \frac{1}{(\Delta t)_c} = B_d$

we say the channel is slow fading

if $B_s \ll (\Delta f)_c$ or $T_s \gg T_m$

we say that the channel is flat.



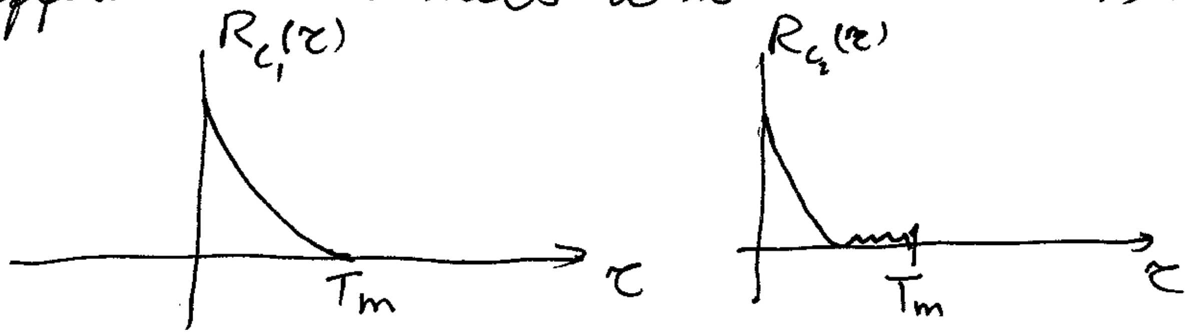


Other measures for assessing the Coherence bandwidth:

We said that

$$(\Delta f)_c \simeq \frac{1}{T_m}$$

However, there might be two completely different channels with the same T_m, e.g.,



A more accurate measure of delay spread is the rms delay spread

$$\sigma_\tau = \sqrt{\overline{\tau^2} - (\bar{\tau})^2}$$

i.e., the variance of τ.

The coherence bandwidth can be approximated as

$$(\Delta f)_c \simeq \frac{1}{50\,\sigma_\tau}$$

if $(\Delta f)_c$ is defined as the range of frequencies over which the channel transfer function has a correlation of 0.9 or more.

A more relaxed assumption, taking the correlation exceeding 0.5, we have

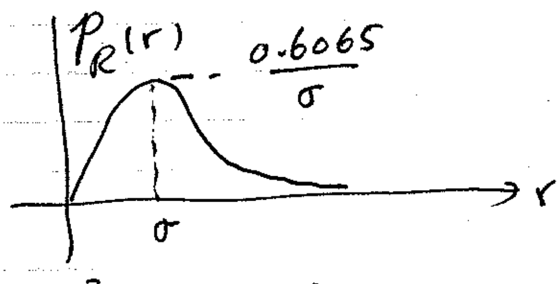
$$(\Delta f)_c \simeq \frac{0.276}{\sigma_\tau}.$$

Rayleigh Fading Channel

If there are a large number of Scatterers in the path between the transmitter and receiver, the channel impulse reponse on each of the two (real and imaginary) axes will be Gaussian. If there is no line of sight (direct) path, then the noise on each axis is zero mean and the ~~amplitude~~ envelope ($r = \sqrt{X_I^2 + X_Q^2}$) has a Rayleigh distribution

$$p_R(r) = \frac{2r}{\Omega} e^{-r^2/\Omega} \qquad r \geq 0$$

where $\Omega = E(R^2)$ is the single parameter needed to characterize the Rayleigh fading. The phase, i.e., $\varphi = \tan^{-1}\left(\frac{n_Q}{n_I}\right)$ will be uniformly distributed on $[-\pi, \pi]$ interval.



$$E[R] = \sigma\sqrt{\frac{\pi}{2}}$$

$$\sigma_R^2 = \sigma^2\left(2 - \frac{\pi}{2}\right)$$

$\Omega = E[R^2] = 2\sigma^2$ where σ^2 is the variance of the Gaussian Components.

Rician Fading Channel:

If there is a line-of-sight (LOS) signal or, in general, a fixed set of scatterers in addition to the random (mobile) ones, we have each of X_I and X_Q as being Gaussian with means m_I and m_Q. Then the distribution is Rice distribution and the channel is called a Rician Fading Channel:

$$p_R(r) = \frac{r}{\sigma^2} e^{-(r^2+s^2)/2\sigma^2} I_0\left(\frac{rs}{\sigma^2}\right) \qquad r \geq 0$$

where $I_0(\cdot)$ is the Bessel function of the first kind and zero-order given by

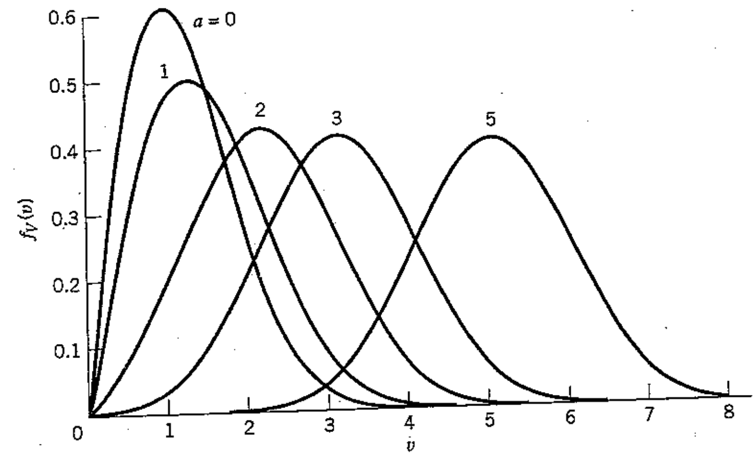
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp[x \cos \alpha]\, d\alpha$$

$I_0(x)$ can be approximated by using a few terms of its expansion:

$$I_0(x) = \sum_{m=0}^{\infty} \left[\frac{(x/2)^m}{m!}\right]^2 .$$

The parameter $s = \sqrt{m_1^2 + m_2^2}$

The ratio $\frac{a^2}{2} = \frac{s^2}{2\sigma^2}$ is called Signal-to-Noise ratio and used to parametrize the Rician distribution. Note that for $a = 0$, the Rician channel becomes Rayleigh channel.



Normalized Rician distribution.

To visualize the Rician Fading channel consider a transmitter sending information to a ~~transmitter~~ receiver where there is a direct path with attenuation α and a single random path with delay $\tau_0(t)$ and attenuation $\beta(t)$. The channel impulse response can be modelled as,

$$c(\tau;t) = \alpha\,\delta(\tau) + \beta(t)\,\delta[\tau - \tau_0(t)]$$

Performance of binary modulation schemes over Rayleigh fading channel:

Take BPSK (the same for QPSK) in a Gaussian Noise channel, we have:

$$P_B = Q\left(\sqrt{2\frac{E_b}{N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right)$$

For a Rayleigh fading channel:

$$\gamma_b = \alpha^2 \frac{E_b}{N_0}$$

Since signal is attenuated by α, the energy is attenuated by α^2.

So:

$$P_B = \int_0^\infty Q\left(\sqrt{2\alpha^2\frac{E_b}{N_0}}\right) p_R(\alpha)\, d\alpha$$

where $p_R(\alpha)$ is the density function of Rayleigh.

It can be shown that for binary PSK:

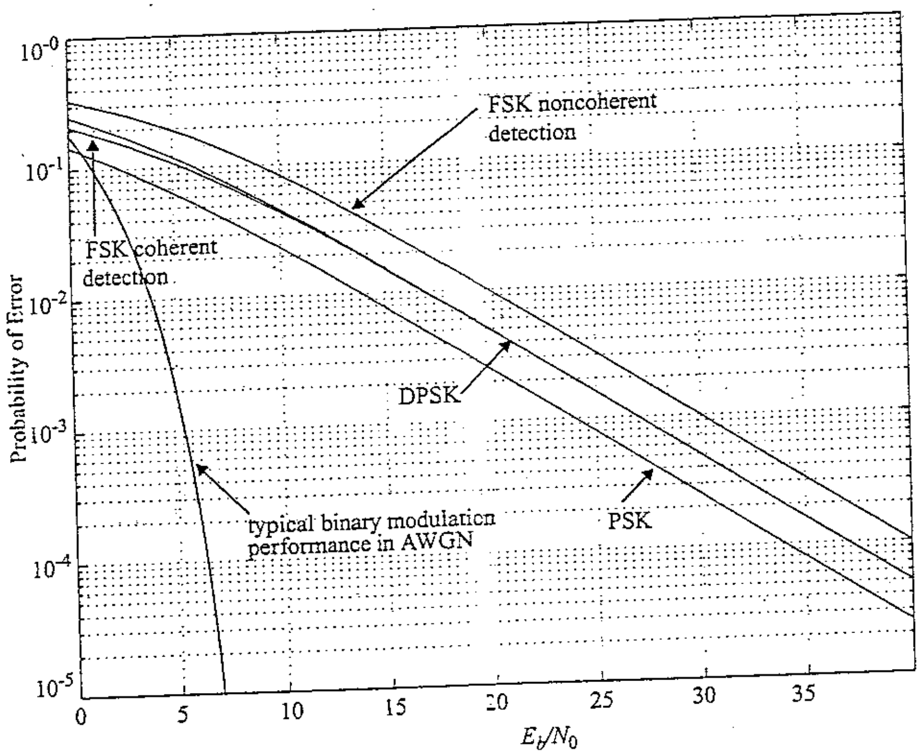
$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_b}{1+\bar{\gamma}_b}}\right)$$

where

$$\bar{\gamma}_b = E[\gamma_b] = \frac{E_b}{N_o} E[\alpha^2]$$

for binary FSK:

$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_b}{2+\bar{\gamma}_b}}\right)$$

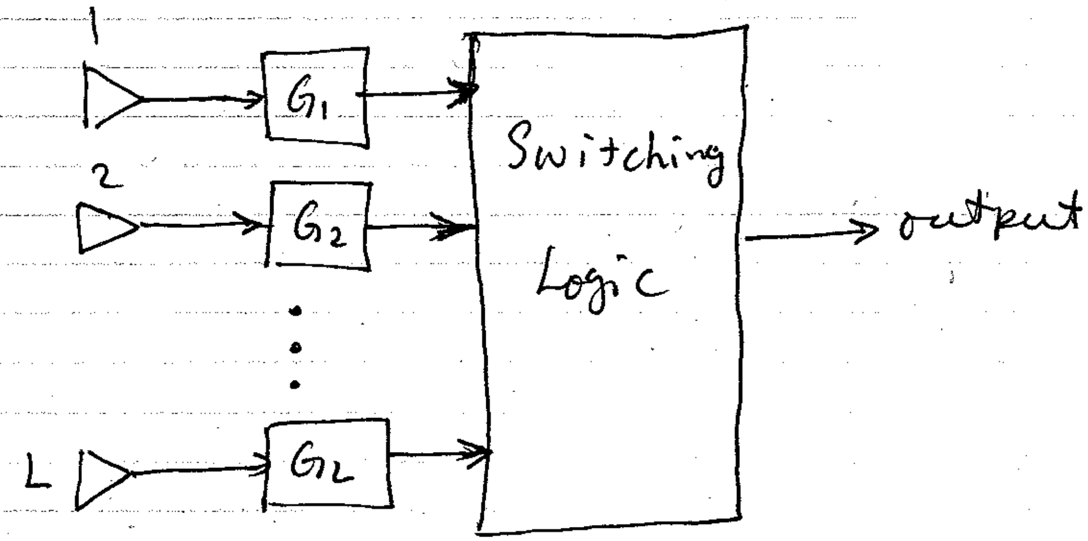


Diversity :

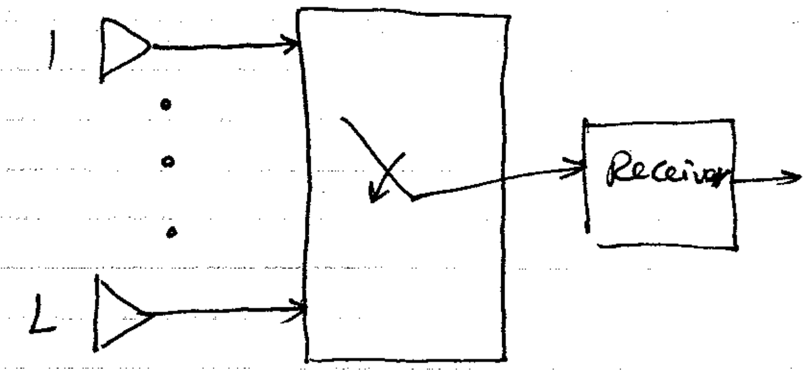
- Frequency Diversity
- Time Diversity
- Spatial (Antenna) Diversity

Diversity Technique :

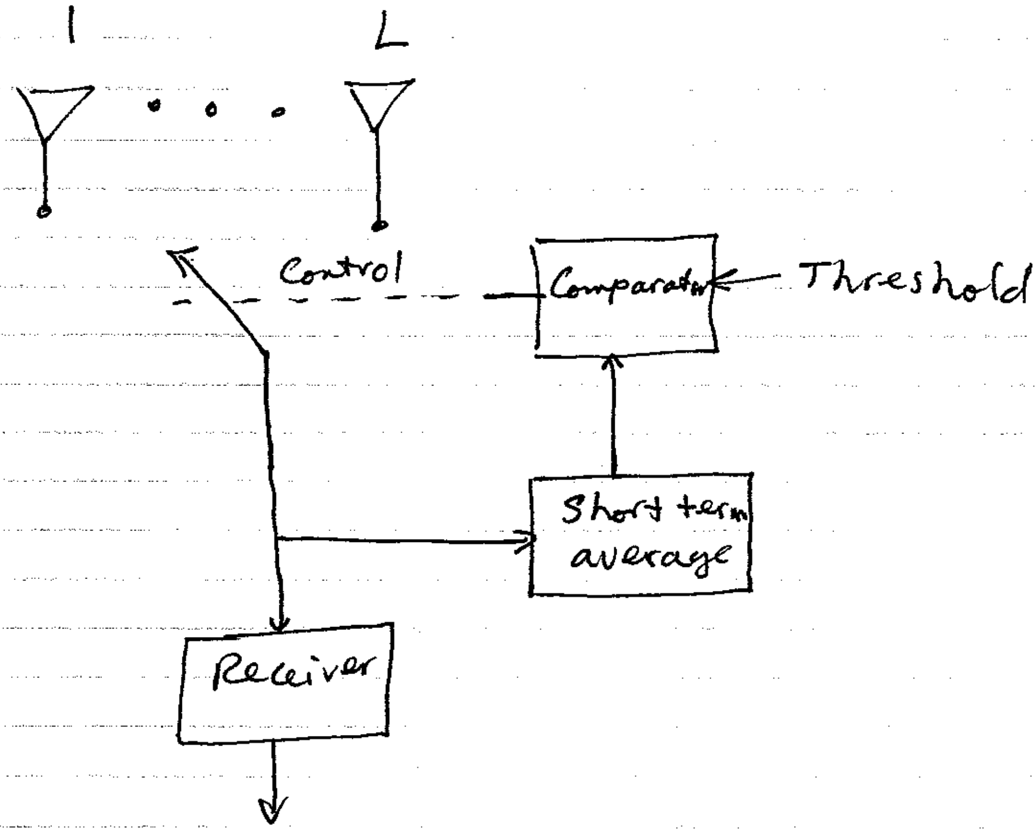
- Selection Diversity
- Feedback (Scanning) Diversity
- Maximal Ratio Combining.
- Equal Gain Combining.



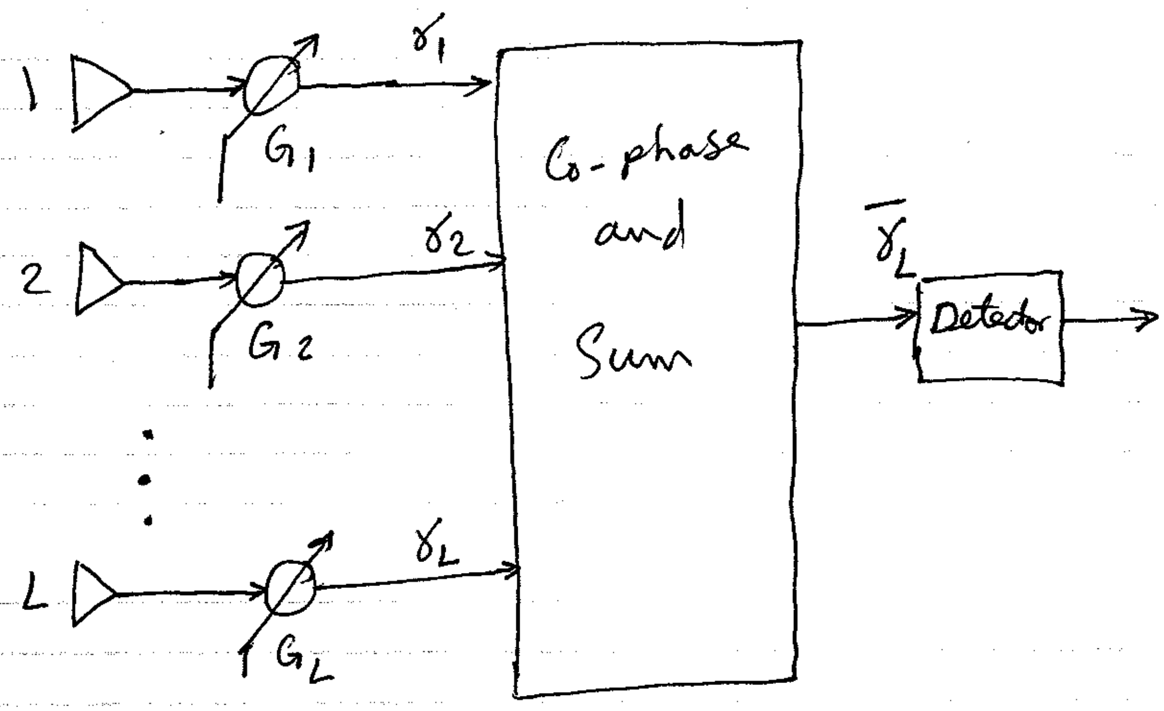
Generalized Block Diagram



Selection Diversity



Scanning (or Feedback) Diversity



Maximal Ratio Combining

Performance of Selection Diversity:

$$\overline{SNR} = \bar{\gamma}_b = \frac{E_b}{N_o} E[\alpha^2]$$

where $\bar{\gamma}_c$ is per branch average SNR

So, the instantaneous SNR, γ_b has density function

$$p_{\gamma_c}(\gamma) = \frac{1}{\bar{\gamma}_b} e^{-\frac{\gamma}{\bar{\gamma}_b}} \qquad \gamma_b \geq 0$$

The probability that SNR is less than a threshold γ is:

$$P(\gamma_b \leq \gamma) = \int_0^{\gamma} p(\gamma_b)\, d\gamma_b = \int_0^{\gamma} \frac{1}{\bar{\gamma}_b} e^{-\frac{\gamma_b}{\bar{\gamma}_b}} d\gamma_b$$

$$= 1 - e^{-\frac{\gamma}{\bar{\gamma}_b}}$$

Now, if we have L independent paths, the probability that SNR of all of them is less than γ is:

$$P_L(\gamma) = P_r(\gamma_1 < \gamma, \gamma_2 < \gamma, \cdots, \gamma_L < \gamma)$$
$$= (1 - e^{-\frac{\gamma}{\bar{\gamma}_c}})^L$$

and the probability that at least one branch has SNR larger than γ is:

$$P(\gamma_b > \gamma) = 1 - P_L(\gamma) = 1 - (1 - e^{-\frac{\gamma}{\bar{\gamma}_c}})^L$$

The pdf of γ_b is:

$$p_L(\gamma_b) = \frac{d}{d\gamma_b} P_L(\gamma_b) = \frac{L}{\bar{\gamma}_c}(1 - e^{-\frac{\gamma_b}{\bar{\gamma}_c}})^{L-1} e^{-\frac{\gamma_b}{\bar{\gamma}_c}}$$

and the average SNR is:

$$\bar{\gamma}_b = \int_0^\infty \gamma\, p_L(\gamma)\, d\gamma = \bar{\gamma}_c \int_0^\gamma L x (1 - e^{-x})^{L-1} e^{-x}\, dx$$

where $x = \frac{\gamma}{\bar{\gamma}_c}$. The above integral can be simplified as,

$$\bar{\gamma}_b = \bar{\gamma}_c \sum_{k=1}^{L} \frac{1}{k}$$

or

$$\frac{\bar{\gamma}_b}{\bar{\gamma}_c} = \sum_{k=1}^{L} \frac{1}{k}$$

The above relationship shows a rapid saturation, i.e., one gets soon to a point of diminishing return.

<u>Example</u>: For a four branch selection diversity scheme, if the average (per branch) SNR is 20 dB, determine the probability that the SNR drops below 10 dB.

Solution: $\bar{\gamma}_c = 100$, $\gamma = 10$

$$\frac{\gamma}{\bar{\gamma}_c} = 0.1 \text{ . So,}$$

$$P_4(10\,dB) = (1 - e^{-0.1})^4 = 8.2 \times 10^{-5}$$

without diversity

$$P_1(10\,dB) = 1 - e^{-0.1} = 0.095 \simeq 10\%$$

X <u>Lecture</u> 13, April 6, 05

Maximal Ratio Combining

Selection Diversity is simple ~~[illegible]~~ to implement since it only requires <u>monitoring</u> and <u>antenna selection switch</u>. It has only one receiver. However, due to the fact that it does not use information from the non-selected branches,

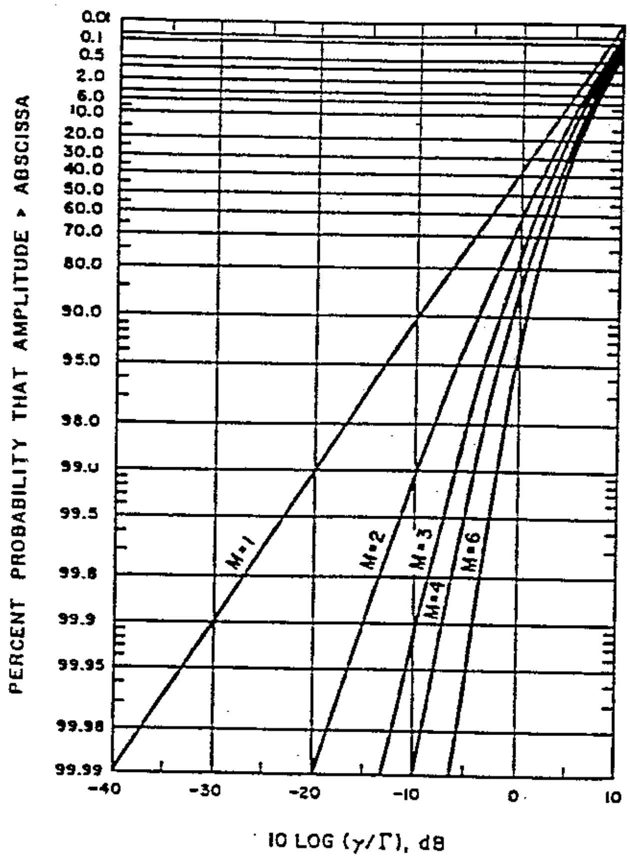


Figure 6.11
Graph of probability distributions of $SNR = \gamma$ threshold for M branch selection diversity. The term Γ represents the mean SNR on each branch [From [Jak71] © IEEE].

it is not making the best use of diversity.

In ratio combining, the voltage signal from L diversity branches are co-phased and added (Coherent Combination), the result is

$$r_L = \sum_{i=1}^{L} G_i r_i$$

where G_i is the gain of the i-th branch.

The received signal power is

$$\frac{r_L^2}{2} = \frac{1}{2}\left[\sum_{i=1}^{L} G_i r_i\right]^2$$

Assume that the noise power is the same in all branches, say, N. Then the total noise N_T is:

$$N_T = N \sum_{i=1}^{L} G_i^2$$

The overall SNR is:

$$\gamma_b = \gamma_L = \frac{r_L^2}{2N_T} = \frac{\left[\sum_{i=1}^{L} G_i r_i\right]^2}{2N\sum_{i=1}^{L} G_i^2}$$

Letting $G_i = k r_i$

$$\gamma_b = \frac{1}{2} \frac{\left(\sum_{i=1}^{L} k r_i^2\right)^2}{N \sum_{i=1}^{L} k^2 r_i^2} = \frac{\left(\sum_{i=1}^{L} r_i^2\right)^2}{N \left(\sum_{i=1}^{L} r_i^2\right)} = \sum_{i=1}^{L} \frac{r_i^2}{2N} = \sum_{i=1}^{L} \gamma_i$$

We note that the SNR of L branches combined is the sum of SNRs of all branches.

The average SNR is

$$\bar{\gamma}_b = \sum_{i=1}^{L} \bar{\gamma}_i = \sum_{i=1}^{L} \bar{\gamma}_c = L \bar{\gamma}_c$$

or

$$\frac{\bar{\gamma}_b}{\bar{\gamma}_c} = L$$

Probability of error for maximal ratio combining when $\bar{\gamma}_c \gg 1$. For coherent PSK (binary) the bit error rate can be approximated as

$$P_b \cong \left(\frac{1}{4\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$

For differential PSK (DPSK, we have

$$P_b \cong \left(\frac{1}{2\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$

For coherent FSK, we have

$$P_{b,FSK} \cong \left(\frac{1}{2\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$

and for non-coherent FSK

$$P_b \cong \left(\frac{1}{\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$

Equations 14.4.38 to 14.4.41 give the probability of symbol error for M-ary PSK.

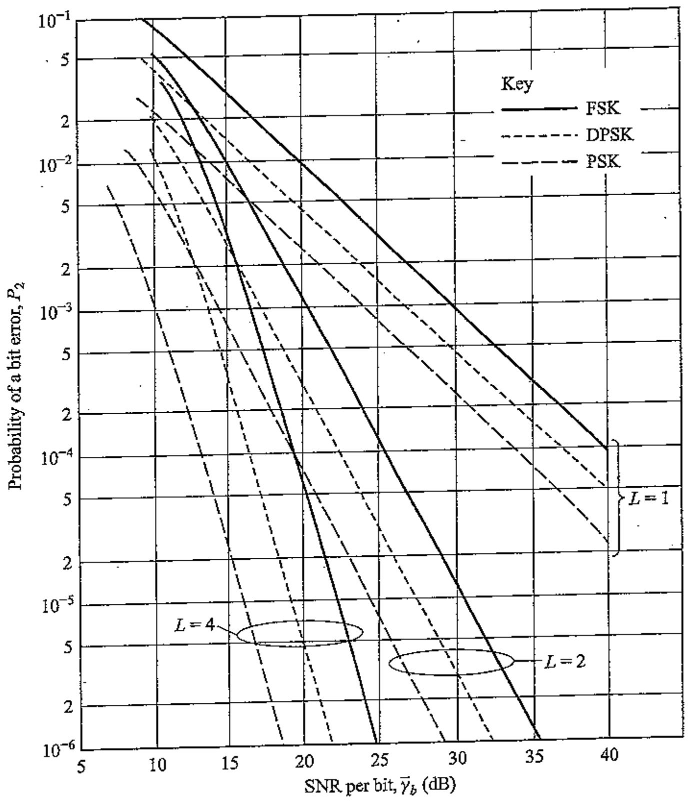


FIGURE 14.4–2
Performance of binary signals with diversity.