Some information theoretic concepts:

Channel Capacity:

For a discrete-memoryless channel, capacity is defined as:

$$C = \max_{p(x)} I(X;Y)$$

where $I(X;Y)$ is the mutual information between the input and the output of the channel:

$$I(X;Y) = H(X) - H(X|Y)$$

where $H(X)$ is the entropy of source with no knowledge of the information at the output of channel and $H(X|Y)$ is the source (input) entropy conditioned on the knowledge of the received signal.

For a Gaussian channel

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right) \quad \text{bits/use}$$

where P is the signal power and N is the noise power.

In terms of rate (in bits/sec.), we have

$$C = W \log\left(1 + \frac{P}{N}\right).$$

C represents the maximum rate at which reliable communication is possible.

Assume that we transmit at a rate of $R = C$.

Then $\quad P = R E_b \quad , N = N_o W \quad$ and $C = R$

$$\frac{R}{W} = \log\left(1 + \frac{E_b}{N_o}\frac{R}{W}\right)$$

$\eta = \frac{R}{W}$ is bandwidth efficiency in bits/sec./Hz.

We get:

$$\frac{E_b}{N_o} = \frac{2^{R/W} - 1}{R/W} = \frac{2^{\eta} - 1}{\eta}.$$

This formula gives the maximum rate (or maximum BW efficiency) for any given $\frac{E_b}{N_o}$.

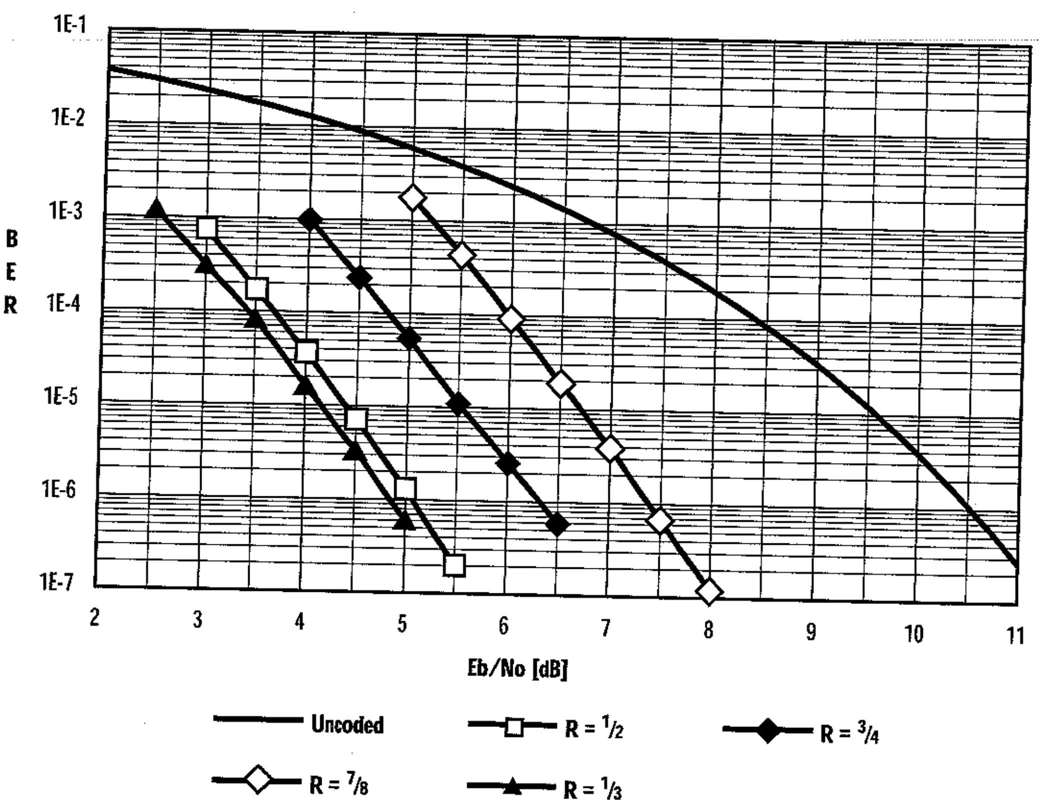
Assume, we want to have $\eta = 1$, i.e., being able to transmit 1 bits/second for each Hertz of bandwidth. Then

$$\frac{E_b}{N_o} = \frac{2^1 - 1}{1} = 1 \Rightarrow \frac{E_b}{N_o} = 0 \text{ dB}.$$

But if we use BPSK: BPSK has an efficiency of 1 bits/sec./Hz., we get BER $= 10^{-5}$ for an $\frac{E_b}{N_0} = 10.5$ dB, i.e., 10.5 dB away from the theory.

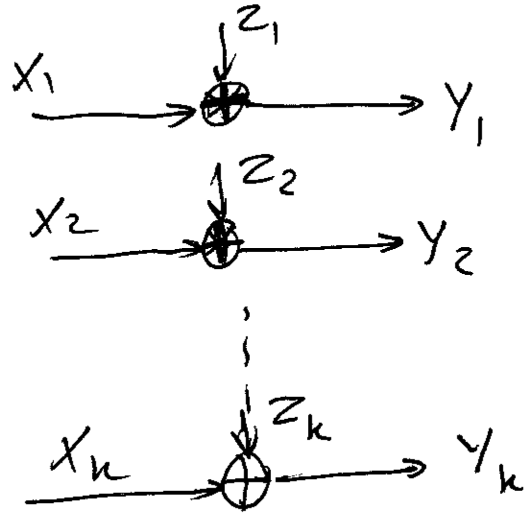
If we use QPSK with rate $\frac{1}{2}$, $K=7$ Convolutional code, we get BER $= 10^{-6}$ at $\frac{E_b}{N_0} = 5$ dB (5.5 dB improvement).

Turbo codes and LDPC codes give much closer results, but for long blocklengths.



Parallel Gaussian Channels:

Assume that we have k parallel channels and a given power that we would like to use to communicate over these channels optimally, i.e., we want to find the optimum allocation of power to these channels:



$$Y_i = X_i + Z_i$$

$$i = 1, 2, \ldots, k$$

$Z_i \sim N(0, N_i)$, i.e., each of the channels add zero-mean Gaussian noise with power N_i.

We would like to find the power of $X_1, \ldots X_k$, i.e., $P_1, \ldots P_k$ subject to constraint

$$\sum_{i=1}^{k} P_i \leq P$$

Total capacity is the sum of the capacities of the individual channels:

$$C = \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right)$$

we would like to maximize C with the constraint $\sum_{i=1}^{k} P_i = P$. and $P_i \geq 0$

We use the method of Lagrange multipliers:

$$J(P_1, P_2, \cdots, P_k) = \sum_{i=1}^{k} \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right) + \mu \sum_{i=1}^{k} P_i$$

Letting $\frac{\partial J}{\partial P_i} = 0$, $i = 1, 2, \cdots, k$

we get

$$\frac{1}{2} \frac{1}{P_i + N_i} + \mu = 0$$

or $P_i = \nu - N_i$

Since P_i's cannot be negative:

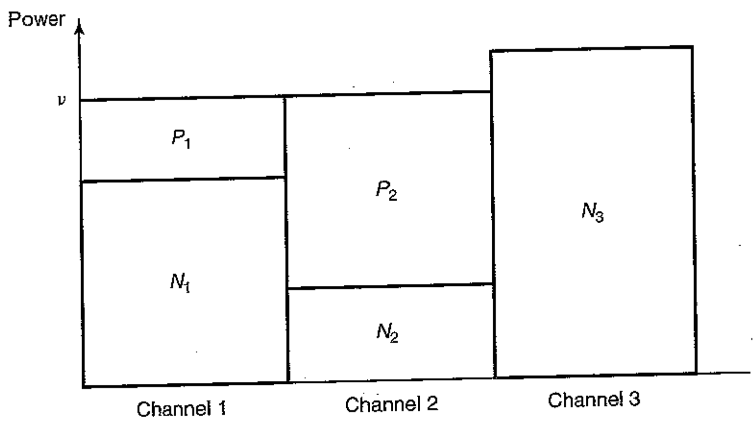
$$P_i = (\nu - N_i)^+ = \begin{cases} \nu - N_i & \nu \geq N_i \\ 0 & \nu < N_i \end{cases}$$

$$= \max(\nu - N_i, 0).$$

where ν is chosen such that:

$$\sum_{i=1}^{k} (\nu - N_i)^{+} = P .$$

This power allocation scheme is called water-filling or water pouring.



Water-filling for parallel channels.

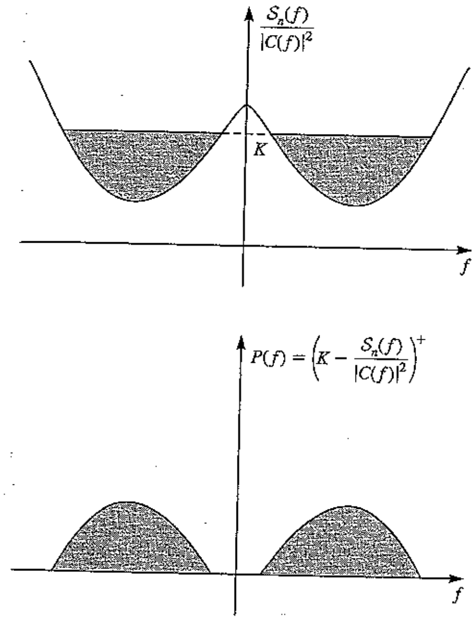
For continuous channels, water pouring scheme results in:

$$C = \frac{1}{2} \int_{-\infty}^{\infty} \log\left(1 + \frac{P(f)\,|C(f)|^2}{S_n(f)}\right) df$$

where $P(f) = \left(\nu - \frac{S_n(f)}{|C(f)|^2}\right)^{+}$

and ν is selected such that

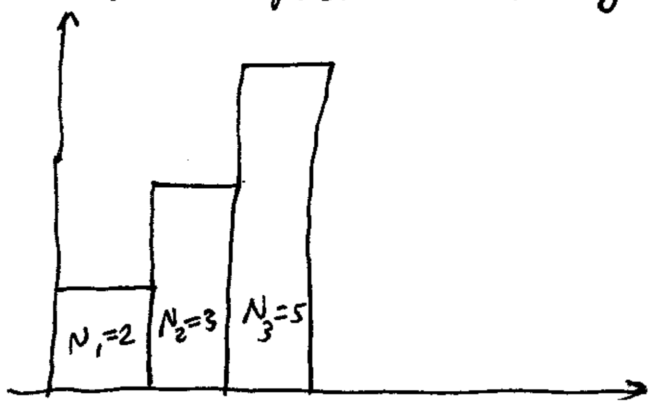
$$\int_{-\infty}^{\infty} P(f)\, df = P$$



The water-filling interpretation of the channel capacity.

Example: Assume that we have three parallel channels with Gaussian noise with powers 2, 3 and 5 mW. Find the power allocation and capacity if:

a) The total available power is 4 mW
b) The total power is 8 mW



a) $P_1 = 2.5$ mW , $P_2 = 1.5$ mW $\quad \nu = 4.5$

$$C = \frac{1}{2}\log\left(1+\frac{2.5}{2}\right) + \frac{1}{2}\log\left(1+\frac{1.5}{3}\right) = 0.9886 \text{ bits/use}$$

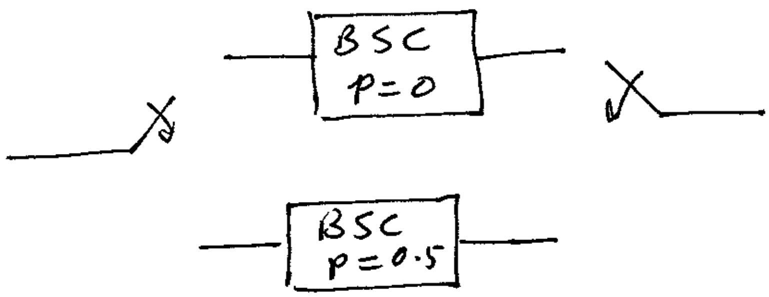
b) $P_1 = 4$ mW, $P_2 = 3$ mW, $P_3 = 1$ mW, $\nu = 6$

$$C = \frac{1}{2}\log\left(1+\frac{4}{2}\right) + \frac{1}{2}\log\left(1+\frac{3}{3}\right) + \frac{1}{2}\log\left(1+\frac{1}{5}\right)$$

$C = 1.481$ bits/~~sec~~

<u>Outage</u> Capacity versus <u>ergodic</u> Capacity:

Consider a channel with two parallel BSC channels (Binary symmetric Channel).



where one of the BSC's are selected at random, the top one with probability δ and the lower one with probability $1-\delta$.

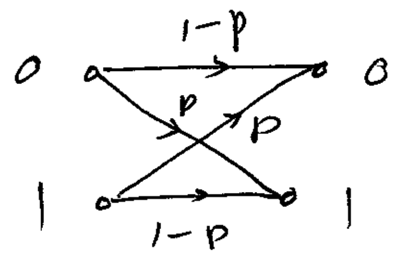
We can consider two situations:

1) At any given instant one of the two BSC's is selected with probabilities δ, $1-\delta$, respectively without any relation to what has happened in the

previous time istant.

2) Once one of the two is selected, the switch stays in its place for the rest of the time.

Each BSC



has capacity:

$$C = 1 - H_b(p) = 1 + p\log_2 p + (1-p)\log_2(1-p)$$

So the capacity of BSC_1 (the top one) is 1 (noiseless transmission) while the capacity of the lower BSC_2 is 0.

In the case 1:

The average (ergodic) capacity is

$$\bar{C} = 1 - H_b\left(\frac{1-\delta}{2}\right).$$

The reason is that:

A zero, e.g., being transmitted goes either through BSC_1, with probability δ

and changes to a one with probability 0.
Or, it goes to BSC_2 with probability $1-\delta$ and gets inverted with probability $\frac{1}{2}$.
The probability of the inversion in two cases is 0 and $\frac{1-\delta}{2}$. So the crossover probability is $\frac{1-\delta}{2}$ and

$$\bar{C} = 1 - H_b\left(\frac{1-\delta}{2}\right).$$

In case 2:

Either BSC_1 is chosen with probability δ. In such a case capacity will be one.
Or, BSC_2 is chosen with probability $1-\delta$ and the average capacity will be set to zero. So,

$$\bar{C} = \delta \times 1 + (1-\delta) \times 0 = \delta$$

Note that while in the first case channel capacity (<u>ergodic</u> capacity) makes sense, i.e., we can say that any rate less than this values is achievable, in the second case what we have found, i.e., δ has no information theoretic significance. In particular, you cannot say anything about rates not exceeding δ. You may transmit at a rate $0 < R \ll \delta$ and still if BSC_2 is selected you get nothing (useful) out of the channel.

In these cases, one can use a concept called <u>outage</u> capacity. We define the outage for a channel as the event of its capacity being less that a given rate, or, as the event of communicating at a rate above its capacity:

$$P_{out}(R) = P[C < R]$$

is called the <u>outage</u> probability.

For a given values $\varepsilon > 0$, the ε-outage capacity is defined as:

$$C_\varepsilon = \max\{R : P_{out}(R) \leq \varepsilon\}$$

For the above channel $C_\varepsilon = \begin{cases} 0 & 0 \leq \varepsilon < 1-\delta \\ \text{[illegible]} & 1-\delta \leq \varepsilon < 1 \end{cases}$

Capacity of Rayleigh fading channels:

1) no CSI

$$C = \max_{p(x)} I(x;y)$$

$$= \max_{p(x)} -\iint p(x)\, p(y|x) \log\left[\frac{p(y|x)}{p(y)}\right] dx\, dy$$

In the case of Rayleigh fading channel

$$p(y|x) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} p(y|x,r,\theta)\, p(r)\, dr\, d\theta$$

where

$$p(r) = \begin{cases} \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}} & r > 0 \\ 0 & r \leq 0 \end{cases}$$

$$p(y|x,r,\theta) = \frac{1}{\pi N_0} e^{\frac{|y - re^{j\theta}x|^2}{N_0}}$$

It can be shown that

$$p(y|x) = \frac{1}{\pi(N_0 + |x|^2)} e^{\frac{-|y|^2}{N_0 + |x|^2}}$$

The computation of C can be done using a parametric formula:

$$P = \mu e^{-\gamma - \psi(\mu)} - 1$$

$$\bar{C} = \frac{\mu - \gamma - \mu\psi(\mu) - 1}{\ln 2} + \log_2 \Gamma(\mu)$$

where

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

and $\gamma = -\psi(1) \approx 0.5772156$.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt .$$

For low SNR, i.e., as $\frac{P}{N_0} \to 0$

$$\bar{C} = \frac{1}{\ln 2} \frac{P}{N_0} \simeq \frac{P}{N_0}$$

2) CSI at the receiver

For this case, since receiver knows phase, it can be compensate for the phase, so we need to find the expected value of capacity for different channel realizations:

$$\bar{C} = E[C] = E[\log(1 + \rho \frac{P}{N_0})]$$

$$\leq \log(1 + E[\rho] \frac{P}{N_0})$$

$$= \log(1 + \frac{P}{N_0})$$

where it is assumed that $E[\rho] = 2\sigma^2 = 1$.

So, the capacity (ergodic capacity) is upper-bounded by the capacity of the AWGN channel.

To find the expression for the capacity:

$$\bar{C} = \int_0^\infty \log(1+\rho\frac{P}{N_o})e^{-\rho}d\rho$$

$$= \frac{1}{\ln 2} e^{\frac{N_o}{P}} \Gamma(0, \frac{N_o}{P})$$

$$= \frac{1}{\ln 2} e^{\frac{1}{SNR}} \Gamma(0, \frac{1}{SNR})$$

where

$$\Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t} dt$$

At low SNR

$$\log(1+\rho\frac{P}{N_0}) \simeq \frac{1}{\ln 2}\frac{P}{N_o}\rho$$

and

$$\bar{C} = \frac{1}{\ln 2}\int_0^\infty \log(\rho\frac{P}{N_o})e^{-\rho}d\rho$$

$$= \log SNR + \frac{1}{\ln 2}\int_0^\infty (\ln\rho)e^{-\rho}d\rho$$

$$= \log SNR - 0.8327$$

(0.8327 less than that of AWGN)

3) State information (CSI) available at both sides:

$$\bar{C} = \int_0^\infty \log\left(1 + \rho \frac{P(\rho)}{N_0}\right) e^{-\rho} d\rho$$

$$\frac{P(\rho)}{N_0} = (1/\rho_0 - 1/\rho)^+$$

and ρ_0 is selected such that

$$\int_0^\infty \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)^+ e^{-\rho} d\rho = \frac{P}{N_0}$$

That is,

$$P(\rho) = \begin{cases} N_0\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) & \rho > \rho_0 \\ 0 & \rho < \rho_0 \end{cases}$$

So:

$$\int_{\rho_0}^\infty \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) e^{-\rho} d\rho = \frac{P}{N_0}$$

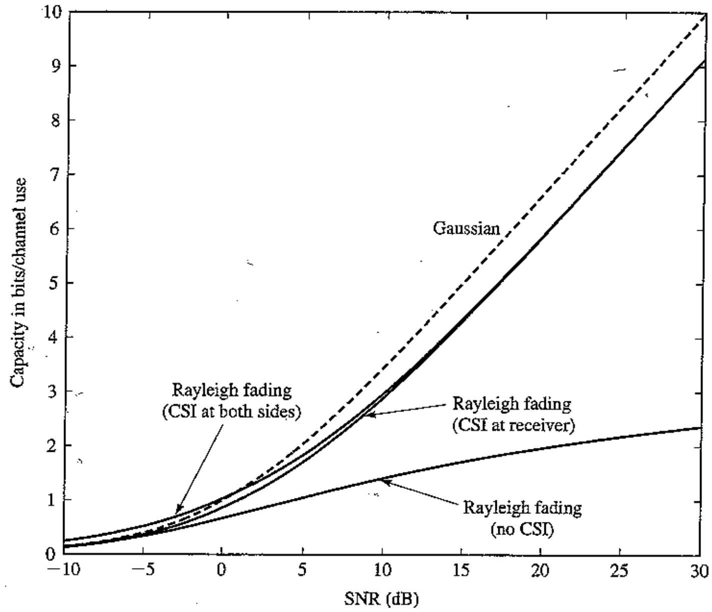
or

$$\frac{e^{-\rho_0}}{\rho_0} - \Gamma(0, \rho_0) = \frac{P}{N_0}$$

Therefore:

$$\bar{C} = \int_{\rho_0}^\infty \left[1 + \rho\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)\right] e^{-\rho} d\rho$$

$$= \int_{\rho_0}^\infty e^{-\rho} \log \frac{\rho}{\rho_0} d\rho = \frac{1}{\ln 2} \Gamma(0, \rho_0)$$



Capacity of Gaussian and Rayleigh fading channel with different CSI.

Outage Capacity of Rayleigh Fading channels:

When ideal interleaving is not possible, e.g., due to restriction on delay (packet length), the system does not experience all channel realizations. In such cases, outage capacity is used instead of ergodic capacity.

For a Rayleigh fading channel

$$P_{out}(R) = P[C < R]$$
$$= P[\log(1 + \rho\, SNR) < R]$$

or

$$P_{out}(R) = P\left[\rho < \frac{2^R - 1}{SNR}\right]$$
$$= 1 - e^{-\frac{2^R - 1}{SNR}} \qquad (A)$$

since

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2} e^{-\rho/2\sigma^2} & \rho > 0 \\ 0 & \rho \leq 0 \end{cases}$$

and we have assumed $2\sigma^2 = 1$.

For high SNR, we will have

$$P_{out}(R) \approx \frac{2^R - 1}{SNR}$$

~~or~~ from (A):

$$R = \log[1 - SNR \ln(1 - P_{out})]$$

So, the ε-outage capacity:

$$C_\varepsilon = \max[R : P_{out}(R) \leq \varepsilon]$$

$$\Rightarrow \boxed{C_\varepsilon = \log[1 - SNR \ln(1-\varepsilon)]}$$

For low SNR

$$C_\varepsilon \simeq \frac{SNR}{\ln 2} \ln \frac{1}{1-\varepsilon}$$

for AWGN channel at low SNR

$$C_{AWGN} = \log(1+SNR) \rightarrow \frac{1}{\ln 2} SNR$$

So,

$$C_\varepsilon = \ln\frac{1}{1-\varepsilon} C_{AWGN}$$

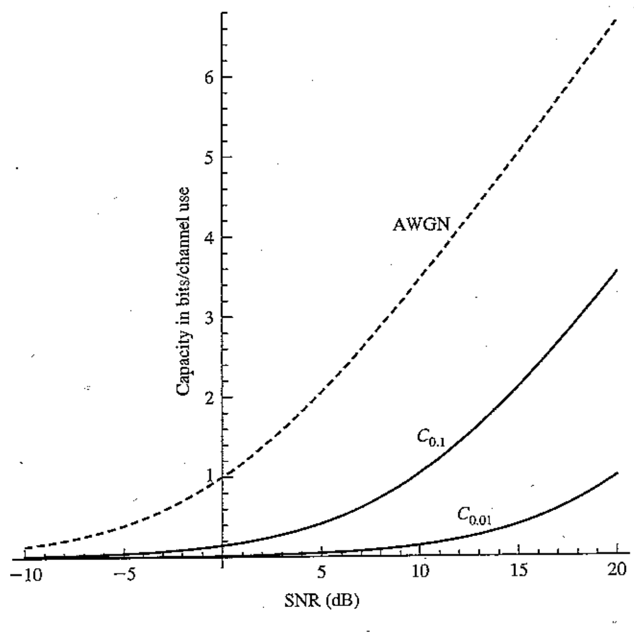
for small ε, $\ln\frac{1}{1-\varepsilon} \rightarrow \varepsilon$ and

$$C_\varepsilon \simeq \varepsilon C_{AWGN}.$$

For high SNR:

$$C_\varepsilon \approx \log\left[SNR \ln\frac{1}{1-\varepsilon}\right]$$

$$= \log SNR + \log\left(\ln\frac{1}{1-\varepsilon}\right) \simeq \log SNR + \log\varepsilon$$



The outage capacity of a Rayleigh fading channel for $\epsilon = 0.1$ and $\epsilon = 0.01$. The capacity of an AWGN channel is given for comparison.

Effect of diversity on Outage Capacity:

When we have a diversity of order L, the random variable $\rho = |R|^2$ is a χ^2 (Chi-square) random varible with $2L$ degrees of freedom. It has a CDF:

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{2\sigma^2}} \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{x}{2\sigma^2}\right)^k & x > 0 \\ 0 & x \leq 0 \end{cases}$$

for $L = 1$ (no diversity), we have an exponential distribution (χ^2 with degrees of freedom of 2)

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2} e^{-\rho/2\sigma^2} & \rho > 0 \\ 0 & \rho \leq 0 \end{cases}$$

For $L \neq 1$, we have

$$P_{out}(R) = P\left[\rho < \frac{2^R - 1}{SNR}\right]$$

$$= 1 - e^{-\frac{2^R-1}{SNR}} \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{2^R-1}{SNR}\right)^k$$

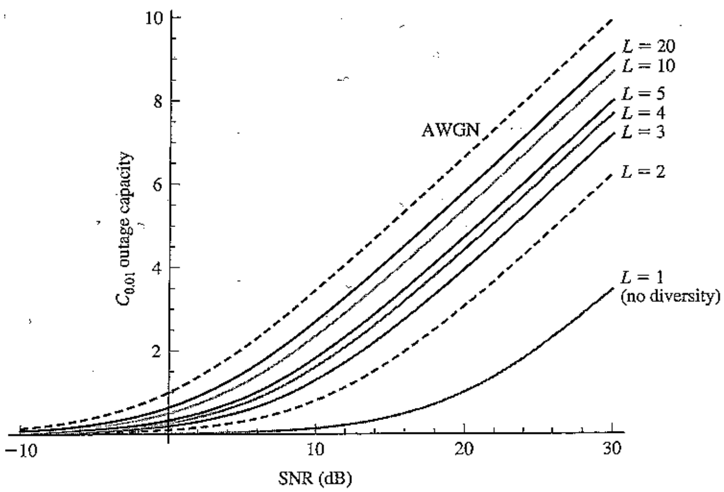
Equating $P_{out}(R)$ ~~to~~ ϵ and solving for R (as C_ϵ), we get:

$$e^{-\frac{2^{C_\varepsilon}-1}{SNR}} \sum_{k=0}^{L-1} \frac{1}{k!}\left(\frac{2^{C_\varepsilon}-1}{SNR}\right) = 1-\varepsilon$$

or

$$e^{-\frac{2^{C_\varepsilon}-1}{SNR}} \sum_{k=L}^{\infty} \frac{1}{k!}\left(\frac{2^{C_\varepsilon}-1}{SNR}\right) = \varepsilon$$

No closed form expression for C_ε exists.
But one can find C_ε approximately using the above relationship.
Following is a plot of $C_{0.01}$ for different L.



The outage capacity of fading channels with different diversity orders.