

X Lecture 4 : May 12, 2011

Baseband Transmission

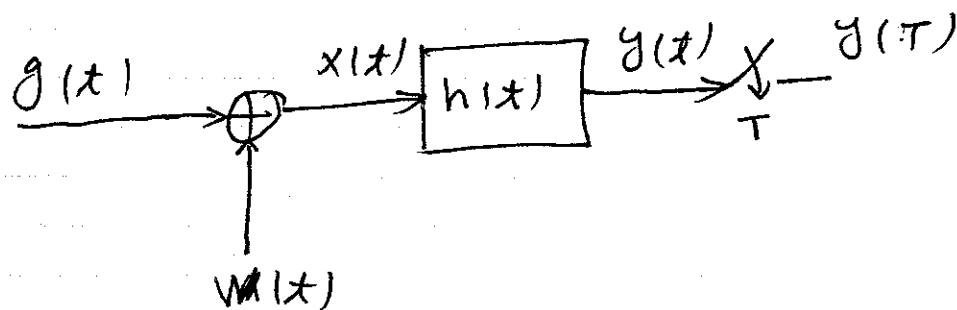
Matched Filter

Assume that a signal $g(t)$ is transmitted in AWGN (Additive White Gaussian Noise) Channel.

Then the received signal is

$$x(t) = g(t) + w(t) \quad 0 \leq t \leq T$$

where T is the signalling interval, i.e., the duration of a signal representing one or more bits.



The received (noisy) signal goes through a receiver filter with impulse response $h(t)$. The output is

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau$$

or

$$y(t) = g_o(t) + n(t)$$

where

$$g_o(t) = \int_{-\infty}^{\infty} g(t-\tau) h(\tau) d\tau$$

and

$$n(t) = \int_{-\infty}^{\infty} w(t-\tau) h(\tau) d\tau$$

Our goal is to maximize the Signal-to-Noise ratio at the output of the receiver filter, i.e.,

to maximize:

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$

$$g_o(t) = \mathcal{F}^{-1}[G(f)] = \mathcal{F}^{-1}[H(f)G(f)]$$

$$= \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft} df$$

So

$$|g_o(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df \right|^2$$

$$\text{and } E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f) df$$

But $S_N(f) = \frac{N_0}{2} |H(f)|^2$

So:

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi f T} df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

We can use Schwartz inequality to find the maximum of η .

Schwarz's inequality

let $\phi_1(x)$ and $\phi_2(x)$ be two functions such that

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty$$

and $\int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty$

then

$$\left| \int_{-\infty}^{\infty} \phi_1(x) \phi_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx$$

with equality if

$$\phi_1(x) = k \phi_2^*(x).$$

So :

$$\left| \int_{-\infty}^{\infty} H(f) G(f) e^{j2\pi fT} df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |G(f)|^2 df$$

So :

$$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df$$

with equality if

$$H(f) = H_{opt}(f) = k G^*(f) e^{-j2\pi fT}$$

or

$$h_{opt}(t) = k \int_{-\infty}^{\infty} G^*(f) e^{-j2\pi fT} \cdot e^{j2\pi ft} df$$

if $g(t)$ is real

$$G^*(f) = G(-f)$$

and

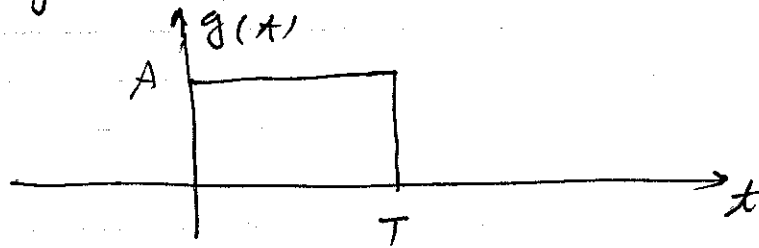
$$h_{opt}(t) = k \int_{-\infty}^{\infty} G(-f) e^{-j2\pi f(T-t)} df$$

$$= k g(T-t)$$

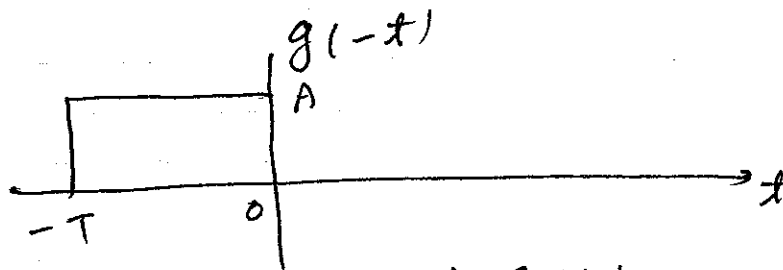
This is called a filter matched to $g(t)$

(a matched filter). This is a time-reversed
and shifted version of $g(t)$.

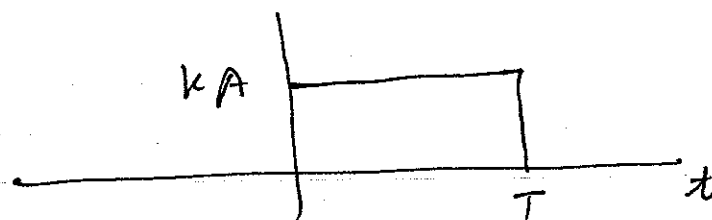
Example : The matched filter for a rectangular pulse



$$h(t) = k g(T-t)$$

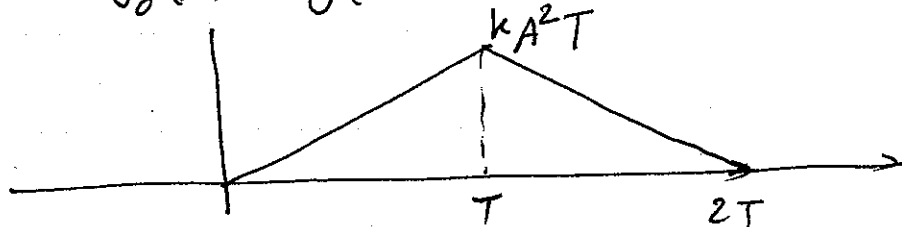


$$g(T-t) = g(t)$$



Matched filter output

$$g_o(t) = g(t) * h(t)$$



Note that

$$\sigma_{\max}^2 = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df = 2 \frac{E}{N_0}$$

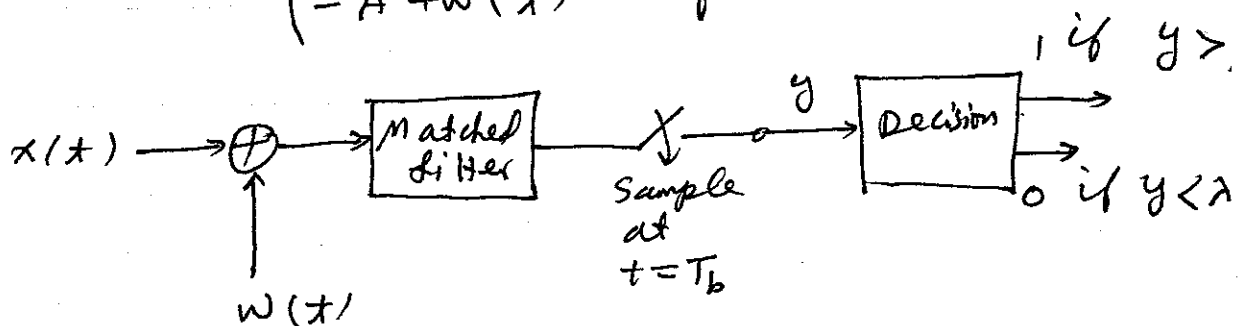
where

$$E = \int_{-\infty}^{\infty} |G(f)|^2 df$$

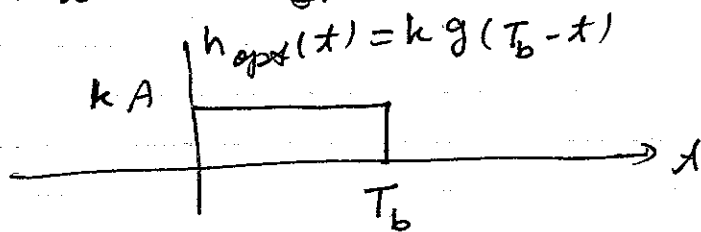
Binary Detection : Probability of error

Assume that we would like to transmit a set of binary values, a stream of bits, over an AWGN channel. Every T_b seconds we transmit a pulse of amplitude A if we have a 1 and a pulse of amplitude of $-A$ to represent a zero. So,

$$x(t) = \begin{cases} A + w(t) & \text{if the bit is 1} \\ -A + w(t) & \text{if the bit is 0.} \end{cases}$$



The matched filter is



let $kAT_b = 1$ (This is just for convenience and does not have any physical significance)

Then

$$y = \int_0^{T_b} x(t) dt = \pm A + \frac{1}{T_b} \int_0^{T_b} w(t) dt$$

y has the mean $\pm A$ depending on whether 0 or 1 was sent. for $\bar{y} = -A$ (when 0 sent)

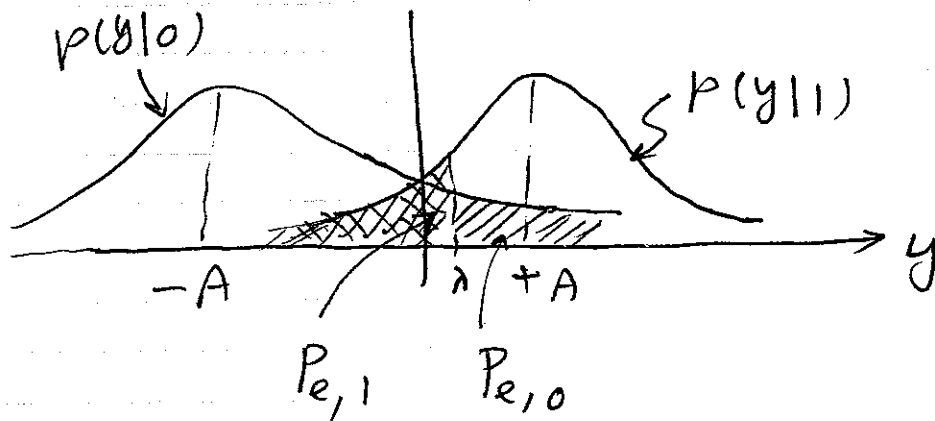
$$\begin{aligned} \sigma_y^2 &= E[(y+A)^2] = \frac{1}{T_b^2} E\left[\int_0^{T_b} \int_0^{T_b} w(t)w(u) dt du\right] \\ &= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} E[w(t)w(u)] dt du \\ &= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} \frac{N_0}{2} \delta(t-u) dt du = \frac{N_0}{2T_b} \end{aligned}$$

So,

$$f_y(y|0) = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[-\frac{(y+A)^2}{N_0/T_b}\right]$$

and

$$f_y(y|1) = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[-\frac{(y-A)^2}{N_0/T_b}\right]$$



$$P_{e,0} = P[\text{error} | 0 \text{ sent}]$$

$$= \int_{\lambda}^{\infty} f_y(y|0) dy$$

$$P_{e,1} = P[\text{error} | 1 \text{ sent}]$$

$$= \int_{-\infty}^{\lambda} f_y(y|1) dy$$

overall Probability of error

$$P_e = P_0 P_{e,0} + P_1 P_{e,1}$$

where P_0 is the probability that a 0 was sent. If $P_0 = P_1 = 1/2$ then it is reasonable to take $\lambda = 0$ and

$$P_e = \frac{1}{2} P_{e,0} + \frac{1}{2} P_{e,1} = P_{e,0} = P_{e,1}$$

We have

$$P_{e,0} = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0/T_b}} e^{-\frac{(y+A)^2}{N_0/T_b}} dy$$

Let $z = \frac{(y+A)}{\sqrt{N_0/2T_b}}$

then $dz = \frac{dy}{\sqrt{\frac{N_0}{2T_b}}}$

and

$$\begin{aligned} P_{e,0} &= \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{\frac{N_0}{2T_b}}}}^{\infty} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{A\sqrt{2T_b}}{\sqrt{N_0}}}^{\infty} e^{-z^2/2} dz \end{aligned}$$

$A^2 T_b = E_b$ energy per bit so:

$$P_{e,0} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E_b}{N_0}}}^{\infty} e^{-z^2/2} dz = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} e^{-z^2/2} dz.$$

So, Probability of error is:

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

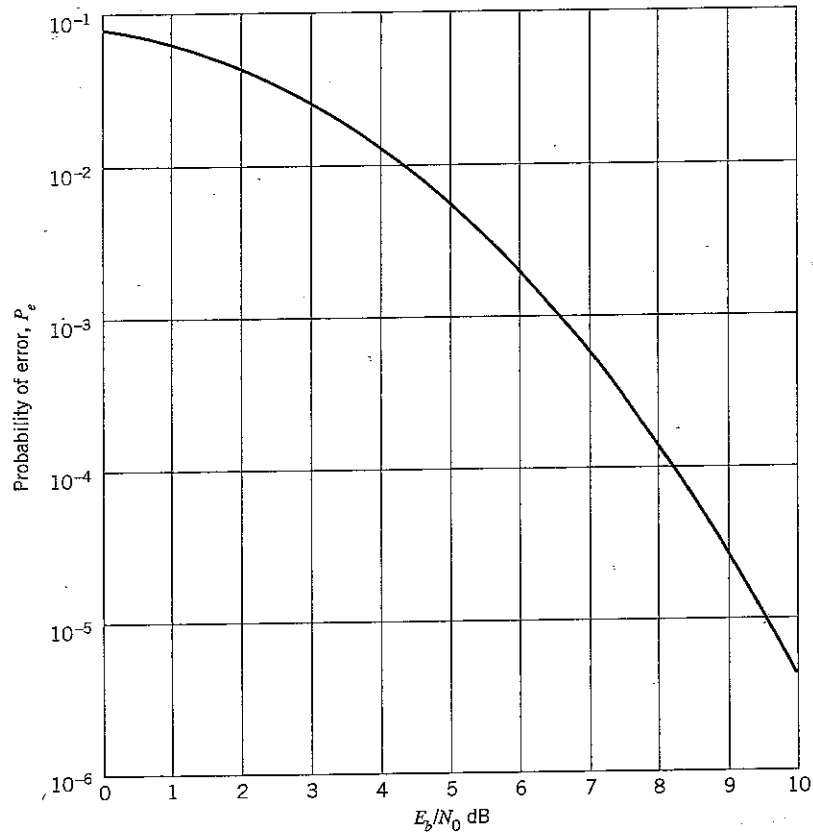


FIGURE 8.7 Probability of error in additive white Gaussian noise with binary signaling.