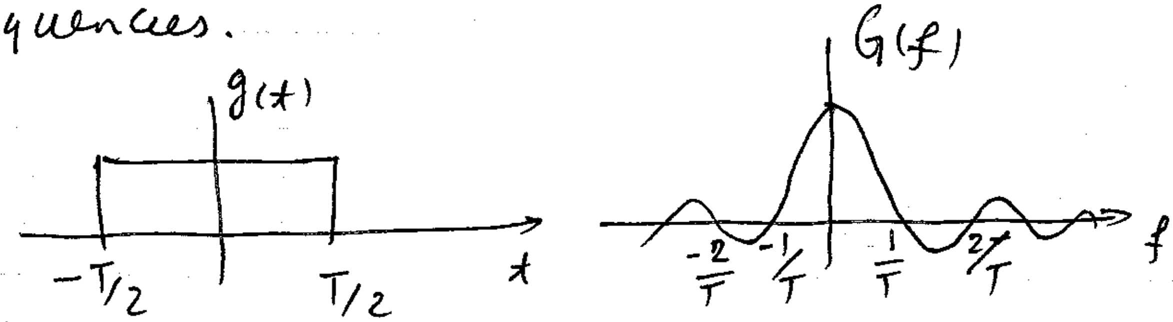
Lecture 5, May 17, 2011

Pulse shaping

So far, we have taken the naive approach of assuming that a 1 is represented by a square pulse and a 0 is represented by its negative.

Note that the Fourier Transform of a square pulse is a Sinc(·) and therefore it extends over all frequencies.

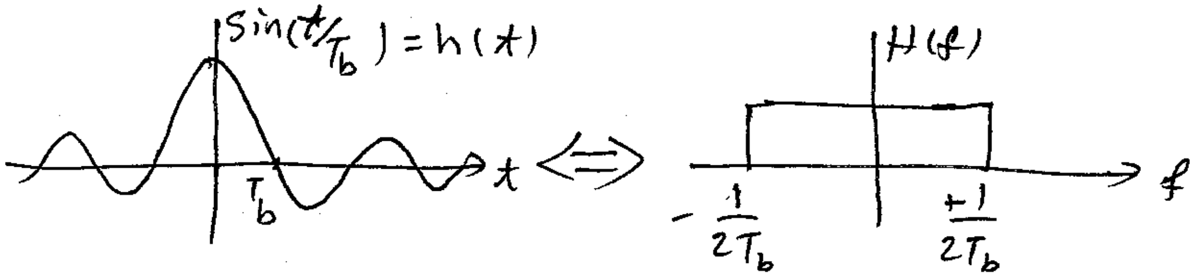


This causes interference between signals transmitted in different channels. To avoid this one needs to use a pulse shape $g(t)$ that has a limited span in frequency domain.

The uncertainty principle, however, indicates that a frequency function which is limited in frequency domain has a time-domain representation that extends over all time. This causes the pulses corresponding to different bits to overlap, causing Inter-Symbol Interference (ISI).

One thing that helps avoiding ISI and still have a spectrum limited in frequency domain is the fact that we are only interested in samples of the received signal at certain points in time, i.e., at $t = 0, T_b, 2T_b$, etc.

So, functions in time-domain that extend over all time, but are zero at kT_b, $k \neq 0$ are useful for pulse shaping. For example, one can use the $\text{Sinc}(\frac{t}{T_b})$ since $\text{Sic}(t/T_b) = 1$ at $t=0$ and is equal to zero when $t = kT_b$, $k=1,2,\cdots$



A new symbol (a new bit in the case of binary) can be sent every T_b seconds, i.e., it is possible to send information at the rate of $R_b = \frac{1}{T_b}$

The require bandwidth is $W = \frac{1}{2T_b} = \frac{R_b}{2}$

This means that the minimum required bandwidth is $\frac{1}{2}$ the bit rate (in general, the symbol rate).

The following figure shows the block diagram of a Communication link

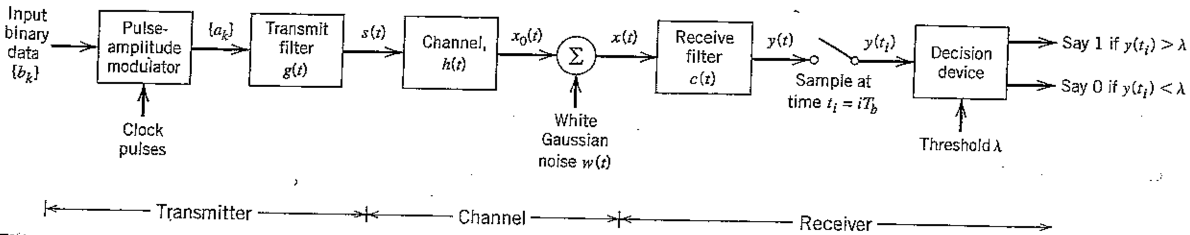


FIGURE 8.8 Baseband binary data transmission system.

At the input every T_b seconds one bit b_k is presented to the system where b_k is either 0 or 1. $\{b_k\}$ is mapped to a stream of levels $\{a_k\}$

$$a_k = \begin{cases} +1 & \text{if } b_k = 1 \\ -1 & \text{if } b_k = 0 \end{cases}$$

The stream of +1 and −1, i.e., $\{a_k\}$ goes through a filter $g(t)$. So, the output $s(t)$ is

$$s(t) = \sum_k a_k g(t - kT_b)$$

$s(t)$ goes through the channel filter $h(t)$ and is corrupted by the channel noise. After filtering at the receiver, the signal $y(t)$ will result:

$$y(t) = \mu \sum_k a_k p(t - kT_b) + n(t)$$

where

$$\mu p(t) = g(t) * h(t) * c(t)$$

$p(t)$ is a normalized pulse, i.e., $p(0) = 1$

and μ is a scaling factor.

In frequency domain,

$$\mu P(f) = G(f) H(f) C(f)$$

Let's now look at the sample of $y(t)$ at

$t_i = iT_b$,

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k p[(i-k)T_b] + n(t_i)$$

$$= \mu a_i + \mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p[(i-k)T_b] + n(t_i)$$

The first term is the desired symbol.

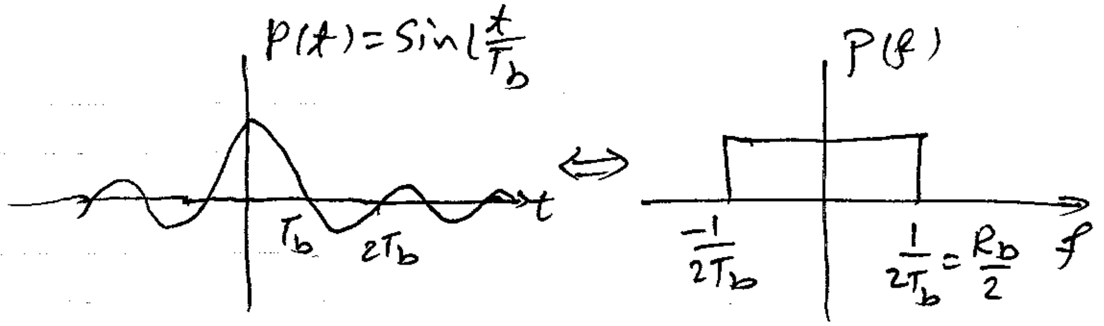
The second term is the ISI

we like the second term to be equal to zero, so that we have $y(t_i) = \mu a_i + n(t_i)$.

As we saw before

$$p(t) = \mathrm{Sinc}\left(\frac{t}{T_b}\right)$$

is a suitable choice resulting (ideally) in zero ISI and $W = \frac{R_b}{2} = \frac{1}{2T_b}$.



The problem with the above pulse (the Nyquist pulse) is:

1) It is not physically realizable. Why? Of course it can be approximated by applying a delay. However, since it decays as $\frac{1}{t}$ (Note that it is $\frac{\sin(\pi t/T)}{\frac{\pi t}{T}}$) the decay is slow and a long delay is needed for approximation to a physically realizable pulse.

2) Due to slow decay, it is sensitive to error in timing, i.e., if instead of sampling at iT_b, we sample at $iT_b + \delta$, then the effect of adjacent symbols is not negligible.

The <u>Nyquist Theorem</u> gives us a tool to find other pulses with better bandwidth-performance tradeoff.

Nyquist Theorem:

In order to have:

$$p(nT_b) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

the Fourier Transform of $p(t)$, i.e., $P(f)$ has to satisfy:

$$\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T_b}\right) = T_b$$

Proof:

$$p(t) = \int_{-\infty}^{\infty} P(f) e^{j2\pi f t} \, df$$

at time $t = nT_b$, we have

$$p(nT_b) = \int_{-\infty}^{\infty} P(f) e^{j2\pi f n T_b} \, df$$

We break the interval $(-\infty, \infty)$ into an infinite interval of duration T_b:

$$p(nT_b) = \sum_{m=-\infty}^{\infty} \int_{\frac{2m-1}{2T_b}}^{\frac{2m+1}{2T_b}} X(f) e^{j2\pi f n T_b} \, df$$

$$p(nT_b) = \sum_{m=-\infty}^{\infty} \int_{-1/2T_b}^{\frac{1}{2T_b}} X\left(f - \frac{m}{T_b}\right) e^{j2\pi f n T_b} \, df$$

$$= \int_{-1/2T_b}^{1/2T_b} \left\{ \sum_{m=-\infty}^{\infty} X\left(f - \frac{m}{T_b}\right) \right\} e^{j2\pi f n T_b} \, df$$

$$= \int_{-1/2T_b}^{1/2T_b} Z(f) \, e^{j2\pi f n T_b} \, df$$

where,

$$Z(f) = \sum_{m=-\infty}^{\infty} X\left(f - \frac{m}{T_b}\right)$$

$Z(f)$ is a periodic function with period $\frac{1}{T_b}$, so, we can represent it using a Fourier Series:

$$Z(f) = \sum_{n=-\infty}^{\infty} z_n \, e^{j2\pi n f T_b}$$

where

$$z_n = T_b \int_{-\frac{1}{2T_b}}^{1/2T_b} Z(f) \, e^{-j2\pi n f T_b} \, df$$

Comparing z_n and $p(nT_b)$ we observe that

$$z_n = T_b \, p(-nT_b)$$

so, in order to have:

$$p(nT_b) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

we have to have:

$$z_n = \begin{cases} T_b & n = 0 \\ 0 & n \neq 0 \end{cases}$$

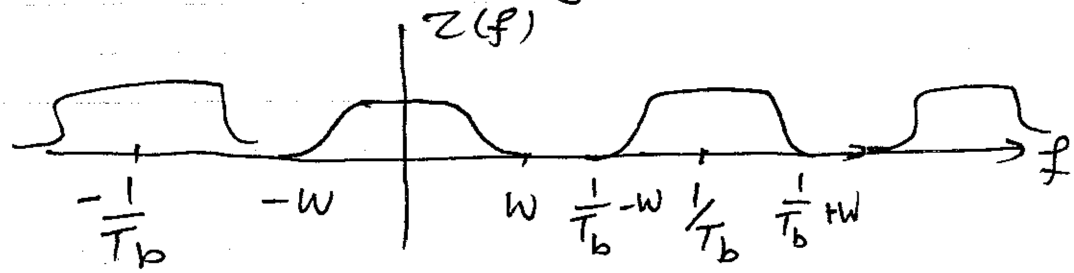
This implies that

$$Z(f) = T_b \qquad \text{for all } f$$

or

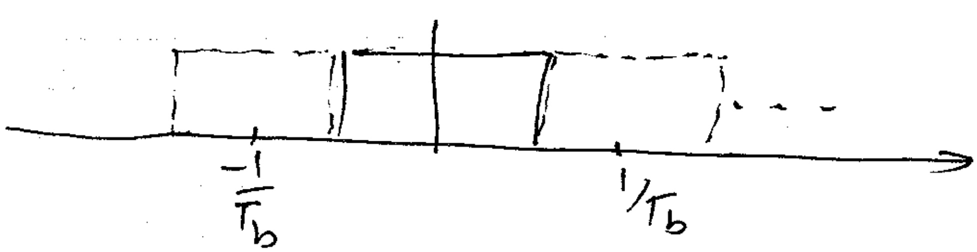
$$\sum_{m=-\infty}^{\infty} P\left(f - \frac{m}{T_b}\right) = T_b \qquad \text{all } f.$$

1) Case $T_b < \frac{1}{2W}$ or $\frac{1}{T_b} = R > 2W$



It is impossible to have zero ISI

2) Case $T_b = \frac{1}{2W}$ or $R_b = 2W$

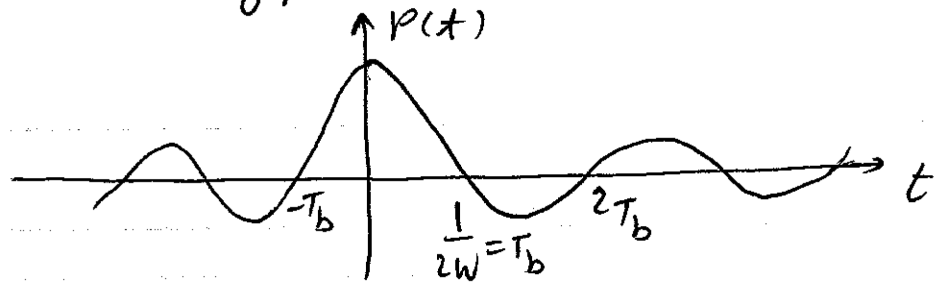


we have to choose:

$$P(f) = \begin{cases} \frac{1}{2W} = T_b & -W < f < W \\ 0 & |f| > W \end{cases}$$

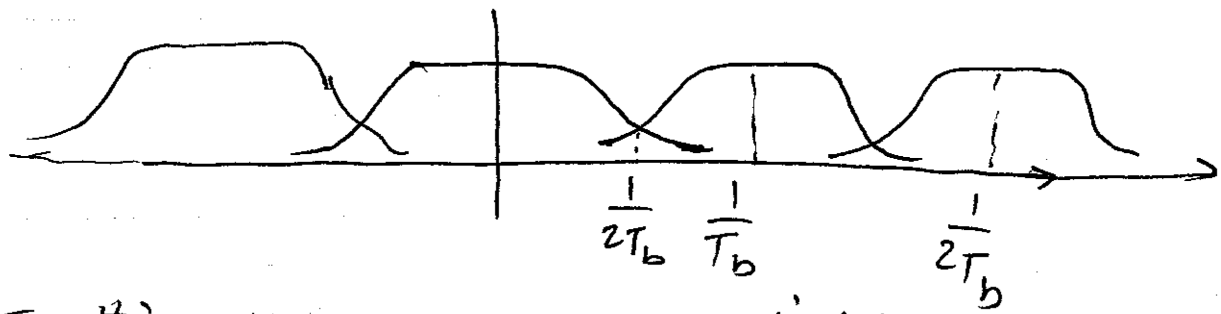
$p(t) = \text{Sinc}\left(\frac{t}{T_b}\right) = \text{Sinc}(2Wt)$

This is the Nyquist pulse discussed above.



Nyquist pulse requires the minimum possible bandwidth. But has the problem of slow decay.

3) $R_b = \frac{1}{T_b} < 2W$ or $T_b > \frac{1}{2W}$



In this case, there are different choices for $P(f)$. The Condition is that $P(f)$ have odd symmetry around $f = \frac{1}{2T_b}$

The spectral characteristic of a raised-cosine filter is given by

$$\mathcal{F}\{f(t)\} = \begin{cases} T, & 0 \le |f| \le \dfrac{1-\beta}{2T} \\ \dfrac{T}{2}\left[1 - \sin \dfrac{\pi T}{\beta}\left(|f| - \dfrac{1}{2T}\right)\right], & \dfrac{1-\beta}{2T} \le |f| \le \dfrac{1+\beta}{2T} \end{cases} \tag{8.3.1}$$

where β is called the *rolloff factor* and can range between 0 and 1. The corresponding time domain Nyquist pulse is

$$f(t) = \frac{\sin \pi t/T}{\pi t/T} \times \frac{\cos \beta\pi t/T}{1 - 4\beta^2 t^2/T^2} \tag{8.3.2}$$

A few examples of raised-cosine frequency responses and their corresponding impulse responses are shown in Fig. 8.8 for selected values of the rolloff parameter β. As can be seen in the figure, small values of β yield the sharpest spectral rolloff characteristics, with $\beta = 0$ corresponding to a rectangular spectrum. The rolloff value $\beta = 1$ eliminates the flat portion of the spectrum and yields a pure raised-cosine shape. Though it is the frequency-domain characteristic which has the raised-cosine characteristic, the corresponding time-domain impulse responses are often called *raised-cosine pulses.*

In practice, a pair of matched filters are used to implement the raised cosine spectrum. The spectrum of the pulse-shaping matched filters used at the transmitter and the receiver is the square root of the raised cosine spectrum. The corresponding impulse response for individual filters is no longer represented by Eq. (8.3.2). This equation represents the overall impulse response used for analysis of the system. For design purposes the impulse response of the square root of the raised cosine function is needed, which is given by

$$f_2(-t) = f_1(t) = \frac{\sin[\pi(1-\beta)t] + 4\beta t \cos[\pi(1+\beta)t]}{\pi\left[1 - (4\beta t)^2\right]t} \tag{8.3.3}$$

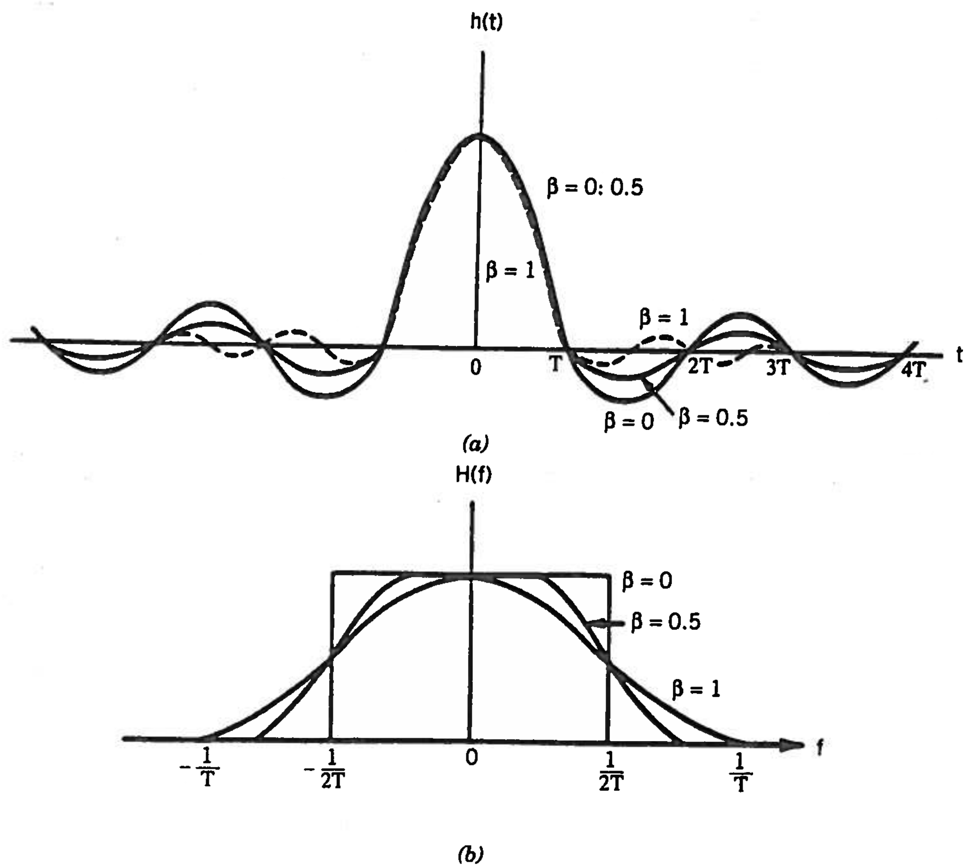
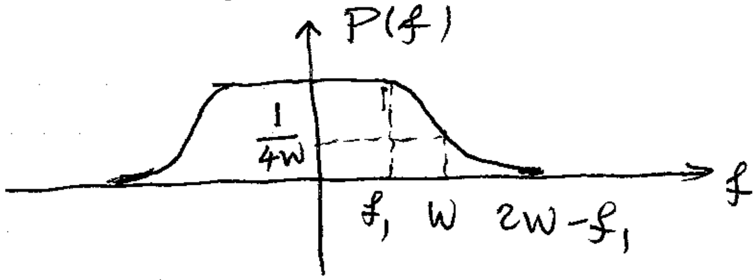


Figure 8.8 Time- and frequency-domain

Raised Cosine filter

One good choice, allowing tradeoff between the bandwidth efficiency and ISI is the raised cosine filter. The filter consists of a flat middle part and a transition part that is basically a sinusoidal shape with a constant added to it. That is why it is calle raised cosine.

$$P(f) = \begin{cases} \frac{1}{2W} & 0 \leq |f| < f_1 \\ \frac{1}{2W}\left\{1 - \sin\left[\frac{\pi(|f|-W)}{2W-f_1}\right]\right. & f_1 \leq |f| < 2W-f \\ 0 & |f| > 2W-f_1 \end{cases}$$

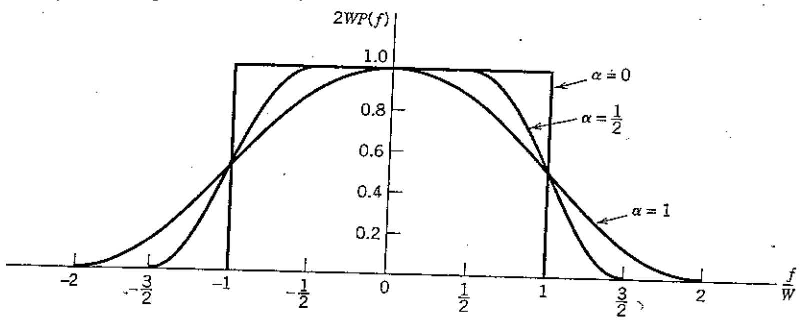


if $f_1 = W$ then the cosine part disappears and we have the Nyquist filter.

The increase in bandwidth compared to the Nyquist pulse is:

$$\alpha = \frac{W-f_1}{W} = 1 - \frac{f_1}{W}$$

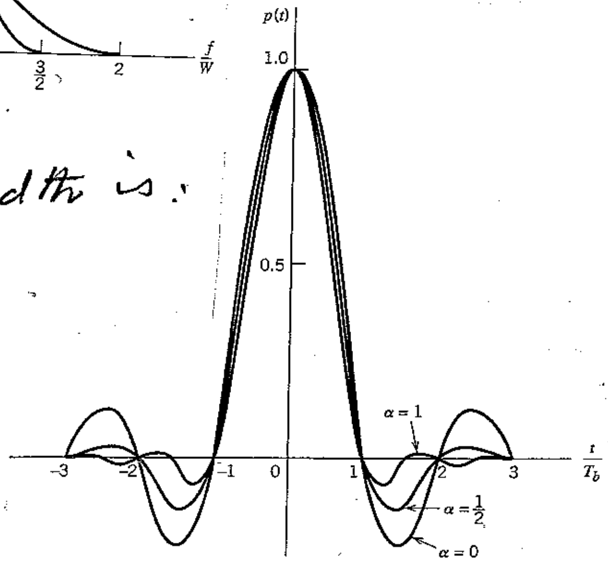
α is called the roll-off factor and is between 0 and 1.



(a)

The required bandwidth is:

$$B = W(1+\alpha) = \frac{1}{2T_b}(1+\alpha)$$



In time domain:

$$p(t) = \mathcal{F}^{-1}[P(f)]$$

$$= \text{Sinc}(2Wt)\frac{\cos(2\pi\alpha W t)}{1-16\alpha^2W^2t^2}$$

or

$$p(t) = \frac{\sin(2\pi W t)\cos(2\pi\alpha W t)}{2\pi W t[1-16\alpha^2W^2t^2]}$$

rate of decay is $\frac{1}{t^3}$ which is much faster than that of a Nyquist pulse.

<u>Example</u>: A raised cosine filter is used for binary transmission of a T_1 signal over a channel with 1 MHz. of bandwidth. What

is the roll-off factor?

$$B = W(1+\alpha) = \frac{R_b}{2}(1+\alpha) = \frac{1.544}{2}(1+\alpha) = 1$$

$$\alpha = 0.295$$

M-ary transmission:

We can send several bits using one symbol. This reduces the required band with. For example, instead of sending $+g(t)$ for 1 and $-g(t)$ for zero. We can send

00	$-3g(t)$
01	$-g(t)$
11	$+g(t)$
10	$+3g(t)$

In this case the symbol rate is half the bit rate, i.e., $R_s = \frac{R_b}{2}$ and the required band width is:

$$B = \frac{R_s}{2}(1+\alpha) = \frac{R_b}{4}(1+\alpha).$$

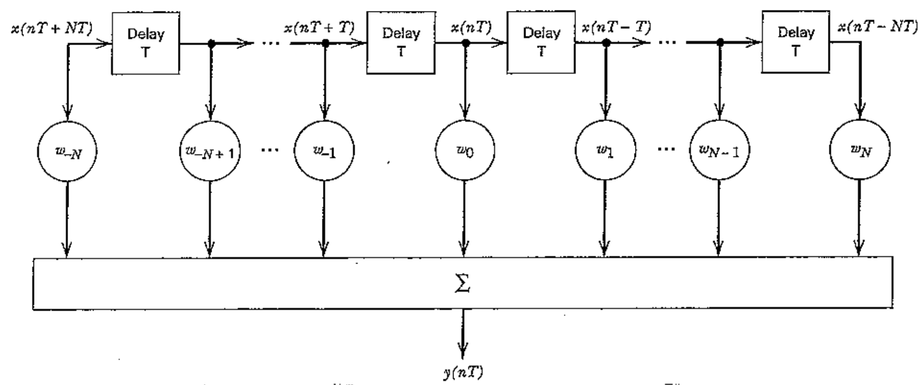
So, we have reduced the required band with by a factor of 2. In general, using

M points we can transmit $\log_2 M$ bits per symbol so, the required bandwidth will be:

$$B = \frac{R_s}{2}(1+\alpha) = \frac{R_b}{2\log_2 M}(1+\alpha)$$

Equalization

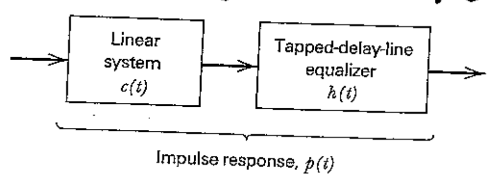
Equalizers are used to mitigate the effect of channel and to reduce (or remove) the ISI. A tapped delay line (or FIR) filter is given as:



The impulse response of this filter is

$$h(t) = \sum_{k=-N}^{N} w_k \, \delta(t-kT)$$

if we place this filter after the channel



The effect will be to produce $p(t)$ as:

$$
\begin{aligned}
p(t) &= c(t) * h(t) \\
&= c(t) * \sum_{k=-N}^{N} w_k \, \delta(t - kT) \\
&= \sum_{k=-N}^{N} w_k \, c(t) * \delta(t - kT) \\
&= \sum_{k=-N}^{N} w_k \, c(t - kT)
\end{aligned}
$$

Sampling at $t_i = iT$, we get

$$p(t_i) = p(iT) = \sum_{k=-N}^{N} w_k \, c[(i-k)T]$$

we would like to have,

$$p(iT) = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

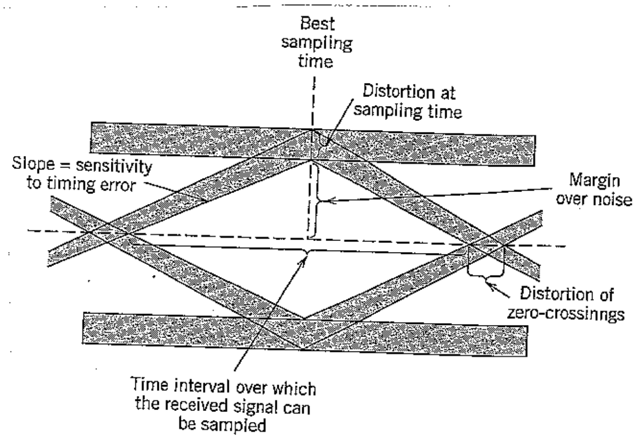
or, we would like

$$\sum_{k=-N}^{N} w_k c_{i-k} = \begin{cases} 1 & i = 0 \\ 0 & i = \pm 1, \pm 2, \pm 3, \cdots \end{cases}$$

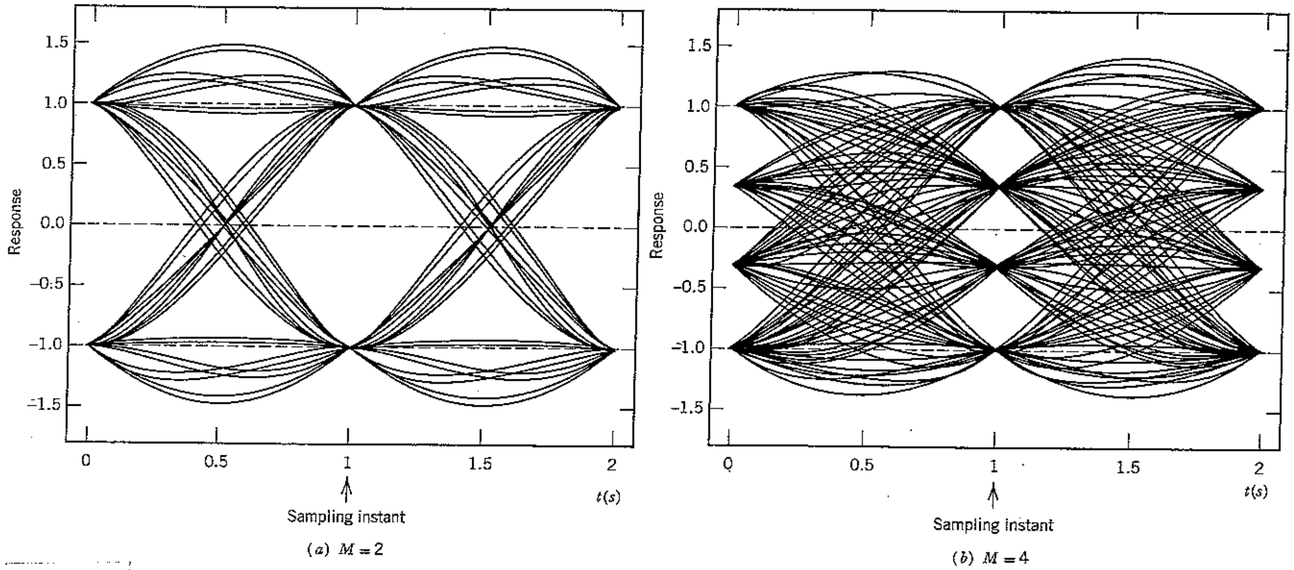
$$
\begin{bmatrix}
c_0 & \cdots & c_{-N+1} & c_{-N} & c_{-N-1} & \cdots & c_{-2N} \\
\vdots & & - & - & - & - & \\
c_{N-1} & & c_0 & c_{-1} & c_{-2} & & c_{-N-1} \\
c_N & & c_1 & c_0 & c_{-1} & & c_{-N} \\
\vdots & & \vdots & \vdots & \vdots & & \\
c_{2N} & & \vdots & \vdots & \vdots & & c_0
\end{bmatrix}
\begin{bmatrix} w_{-N} \\ \vdots \\ w_{-1} \\ w_0 \\ \vdots \\ w_N \end{bmatrix}
=
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Eye Diagram

Eye Diagrams are use for assessing the quality of transmission and the sensitivity of the link to timing jitter and noise. An eye diagram is formed on an ossiloscope by triggering the ossiloscope with the (recovered) timing signal and feeding the received signal to the vertical of the oscilloscope the result will be the superimposition of the different waveshapes (corresponding to adjacent bits). What we see on the scope looks like an eye.



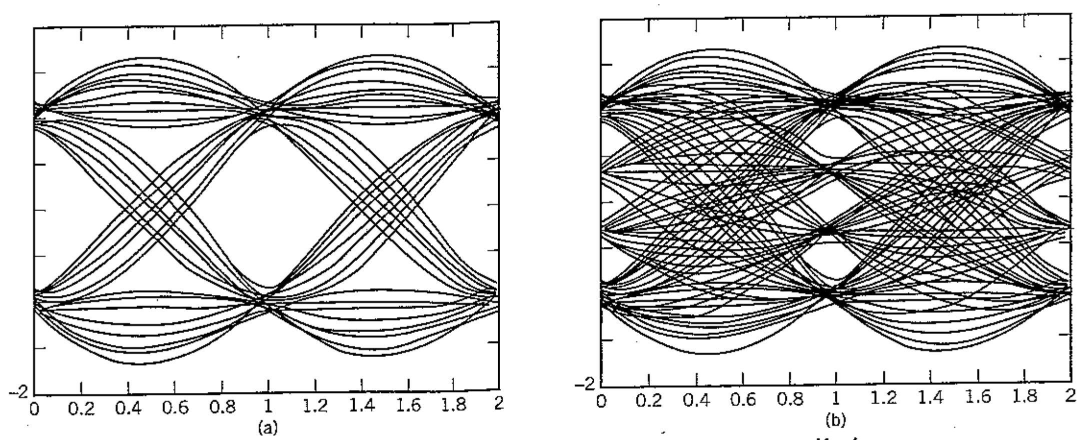
The more open the eye is, i.e., less jitter and other distortions, the more noise margin we have. Following are the typical eye diagrams for binary and quaternary ($M=4$) transmission.



(a) $M = 2$ (b) $M = 4$

Eye diagram of received signal with no bandwidth limitation.

When the bandwidth is smaller than what is required (rate is more than it should be) the eye narrows and, in sever cases, closes.



(a) $M = 2$ (b) $M = 4$

Eye diagram of received signal using a bandwidth-limited channel response.