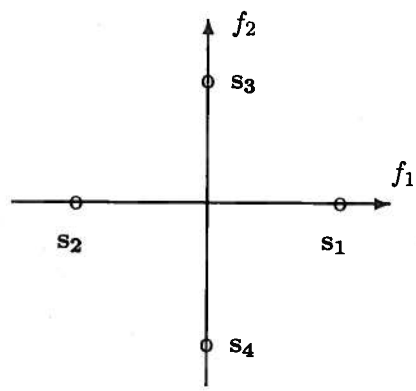
Problem 3.1

Assuming M is even we have

$$
\begin{aligned}
1^2+3^2+5^2+\ldots(M-1)^2 &= \sum_{i=1}^{M} i^2 - \sum_{k=1}^{M/2}(2k)^2 \\
&= \frac{M(M+1)(2M+1)}{6} - 4\sum_{k=1}^{M/2} k^2 \\
&= \frac{M(M+1)(2M+1)}{6} - 4\frac{M/2(M/2+1)(M+1)}{6} \\
&= \frac{M(M^2-1)}{6}
\end{aligned}
$$

Problem 3.2

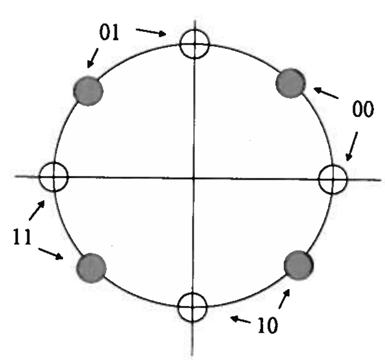
$$
\begin{aligned}
\mathbf{s}_1 &= \left(\sqrt{\mathcal{E}}, 0\right) \\
\mathbf{s}_2 &= \left(-\sqrt{\mathcal{E}}, 0\right) \\
\mathbf{s}_3 &= \left(0, \sqrt{\mathcal{E}}\right) \\
\mathbf{s}_4 &= \left(0, -\sqrt{\mathcal{E}}\right)
\end{aligned}
$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

3.3

1.2. The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT \quad n = 0, 1, \ldots$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1, \quad n = 0, 1, \ldots$ are given with filled circles.

Problem 3.4

1. Consider the QAM constellation of Fig. P3-4. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

2. If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

3. The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

Problem 3.5

1. Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P3-4. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$\text{B} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad \text{C} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2. Since each symbol conveys 3 bits of information, the resulted symbol rate is :

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

Problem 3.6

The constellation of Fig. P3-6(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is:

$$P_a = \frac{1}{8}\left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2\right] = 6A^2$$

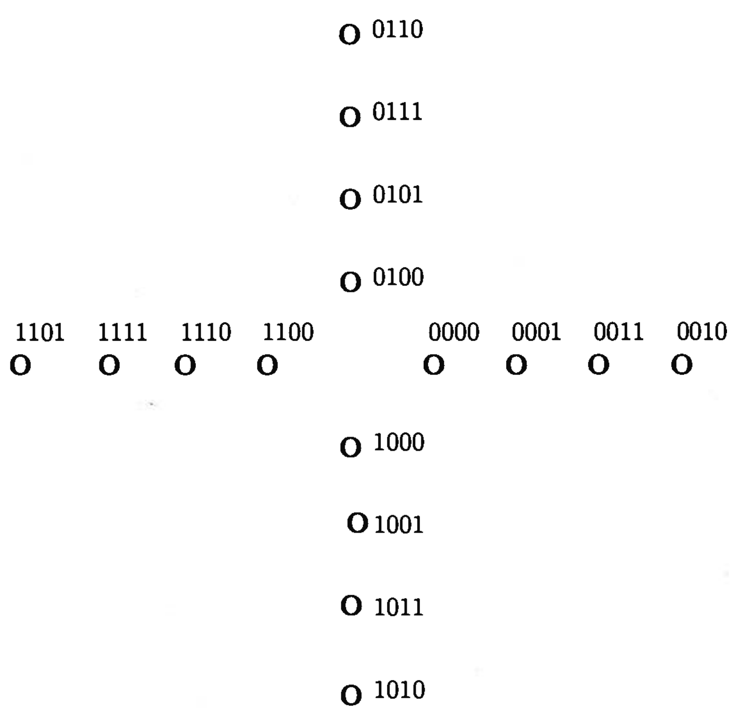
The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is:

$$P_b = \frac{1}{8}\left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$ 5

Since $P_b < P_a$ the second constellation is more power efficient.

Problem 3.7

One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure.



Problem 3.21

1) The power spectral density of $X(t)$ is given by

$$\mathcal{S}_x(f) = \frac{1}{T}\mathcal{S}_i(f)|U(f)|^2$$

The Fourier transform of $u(t)$ is

$$U(f) = \mathcal{F}[u(t)] = AT\frac{\sin \pi f T}{\pi f T}e^{-j\pi f T}$$

Hence,

$$|U(f)|^2 = (AT)^2 \text{sinc}^2(fT)$$

and therefore,

$$\mathcal{S}_x(f) = A^2 T \mathcal{S}_i(f) \text{sinc}^2(fT) = A^2 T \text{sinc}^2(fT)$$

2) If $u_1(t)$ is used instead of $u(t)$ and the symbol interval is T, then

$$\begin{aligned} \mathcal{S}_x(f) &= \frac{1}{T}\mathcal{S}_a(f)|U_1(f)|^2 \\ &= \frac{1}{T}(A2T)^2 \text{sinc}^2(f2T) = 4A^2 T \text{sinc}^2(f2T) \end{aligned}$$

3) If we precode the input sequence as $b_n = I_n + \alpha I_{n-1}$, then

$$R_b(m) = \begin{cases} 1+\alpha^2 & m = 0 \\ \alpha & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the power spectral density $\mathcal{S}_b(f)$ is

$$\mathcal{S}_b(f) = 1 + \alpha^2 + 2\alpha \cos(2\pi f T)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter α should be such that

$$1 + \alpha^2 + 2\alpha \cos(2\pi f T)|_{f=\frac{1}{3T}} = 0$$

and α does not have a real-valued solution. Therefore the above precoding cannot result in a PAM system with the desired spectral null.

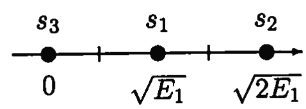
4) The answer to this question is no. This is because $\mathcal{S}_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 4.5

1. Note that $s_2(t) = 2s_1(t)$ and $s_3(t) = 0s_1(t)$, hence the system is PAM and a singular basis function of the form $\phi_1(t) = \frac{1}{A\sqrt{T}}s_1(t)$ would work

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 < t \leq T/3 \\ -\frac{1}{\sqrt{T}} & T/3 \leq t < T \end{cases}$$

Assuming $E_1 = A^2T$, we have $s_3 = 0$, $s_1 = \sqrt{E_1}$, $s_2 = 2\sqrt{E_1}$. The constellation is shown below.



2. For equiprobable messages the optimal decision rule is the nearest neighbor rule and the perpendicular bisectors are the boundaries of the decision regions as indicated in the figure.

3. This is ternary PAM system with the distance between adjacent pints in the constellation being $d = \sqrt{E_1} = A\sqrt{T}$. The average energy is $E_{\text{avg}} = \frac{1}{3}(0 + A^2T + 4A^2T) = \frac{5}{3}A^2T$, and $E_{\text{bavg}} = E_{\text{avg}}/\log_2 3 = \frac{5}{3\log_2 3}A^2T$, from which we obtain

$$d^2 = \frac{3\log_2 3}{5}E_{\text{bavg}} \approx 0.951E_{\text{bavg}}$$

The error probability of the optimal detector is the average of the error probabilities of the three signals. For the two outer signals error probability is $P(n > d/2) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$ and for the middle point s_1 it is $P(|n| > d/2) = 2Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$. From this,

$$P_e = \frac{4}{3}Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{0.951E_{\text{bavg}}}{2N_0}}\right) = 43Q\left(\sqrt{0.475\frac{E_{\text{bavg}}}{2N_0}}\right)$$

4. $R = R_s \log_2 M = 3000 \times \log_2 3 \approx 4755$ bps.

Problem 4.6

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 4.7

1. The PDF of the noise n is :

$$p(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

where $\lambda = \frac{\sqrt{2}}{\sigma}$ The optimal receiver uses the criterion :

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

The average probability of error is :

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{\frac{-\sqrt{2}A}{\sigma}}
\end{aligned}$$

/

2. The variance of the noise is :

$$\begin{aligned}\sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\ &= \lambda\int_0^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2\end{aligned}$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2SNR}}$$

For $P(e) = 10^{-5}$ we obtain:

$$\ln(2 \times 10^{-5}) = -\sqrt{2SNR} \Longrightarrow SNR = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{SNR}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{SNR} = 4.26$ and therefore SNR $= 18.1476 = 12.594$ dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 4.8

1. Since $d_{\min} = 2A$, from the union bound we have $P_e \leq 15Q\left(\sqrt{d_{\min}^2/2N_0}\right) = 15Q\left(\sqrt{2A^2/N_0}\right)$.

2. Three levels of energy are present, $E_1 = A^2 + A^2 = 2A^2$, $E_2 = A^2 + 9A^2 = 10A^2$, and $E_3 = 9A^2 + 9A^2 = 18A^2$. The average energy is $E_{\text{avg}} = \frac{1}{4}E_1 + \frac{1}{2}E_2 + \frac{1}{4}E_3 = 10A^2$. Therefore, $E_{\text{bavg}} = E_{\text{avg}}/\log_2 16 = 2.5A^2$.

3. $P_e \leq 15Q\left(\sqrt{2A^2/N_0}\right) = 15Q\left(\sqrt{4E_{\text{bavg}}/5N_0}\right)$.

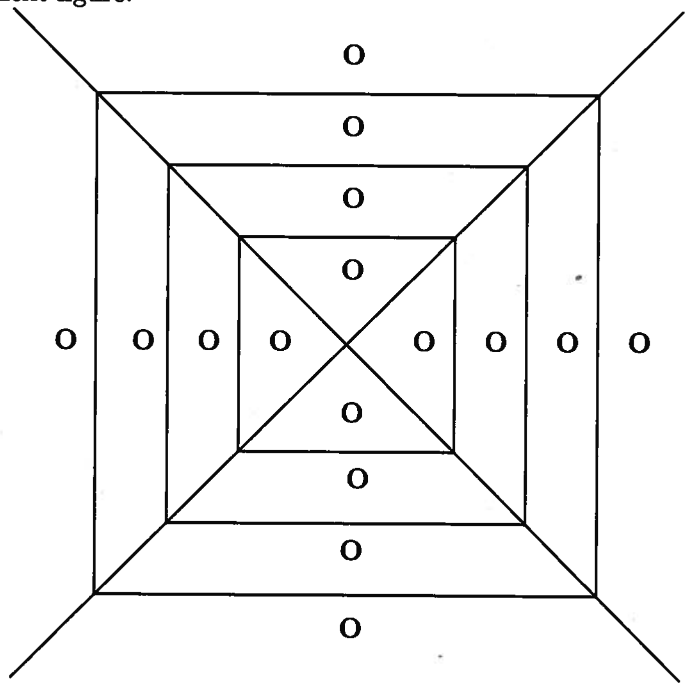
4. For a 16-level PAL system

$$P_e \approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_{\text{bavg}}}{N_0}}\right) = 2Q\left(\sqrt{\frac{24}{255}\frac{E_{\text{bavg}}}{N_0}}\right)$$

The difference is $\frac{4/5}{24/255} = 255/30 \approx 8.5 \sim 9.3$ dB

Problem 4.20

The optimum decision boundary of a point is determined by the perpedicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:



Problem 4.28

Using the Pythagorean theorem for the four-phase constellation, we find:

$$r_1^2 + r_1^2 = d^2 \Longrightarrow r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus:

$$d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^o) \Longrightarrow r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}$$

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

$$P_{4,av} = \frac{d^2}{2}, \qquad P_{8,av} = \frac{d^2}{2-\sqrt{2}}$$

Thus, the additional transmitted power needed by the 8-PSK signal is:

$$P = 10\log_{10}\frac{2d^2}{(2-\sqrt{2})d^2} = 5.3329 \text{ dB}$$

We obtain the same results if we use the probability of error given by (see 4-3-17) :

$$P_M = 2Q\left[\sqrt{2\gamma_s}\sin\frac{\pi}{M}\right]$$

where γ_s is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

$$\gamma_{4,s}\sin^2\frac{\pi}{4} = \gamma_{8,s}\sin^2\frac{\pi}{8} \Longrightarrow 10\log_{10}\frac{\gamma_{8,s}}{\gamma_{4,s}} = 20\log_{10}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}} = 5.3329 \text{ dB}$$

Since we consider that error ocuur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance d between adjacent points.

Problem 4.44

1. Using the definition of Q-function $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$ we have

$$\begin{aligned} E\left[Q(\beta X)\right] &= \int_0^\infty Q(\beta x) \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\, dx \\ &= \int_0^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{\beta x}^\infty e^{-\frac{t^2}{2}}\, dt\right] \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\, dx \end{aligned}$$

where the region of integration is $0 \le x < \infty$ and $\beta x \le t < \infty$. Reordering integrals we have

$$\begin{aligned} E\left[Q(\beta X)\right] &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \left[\int_0^{\frac{t}{\beta}} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\, dx\right] dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left(1 - e^{-\frac{t^2}{2\beta^2\sigma^2}}\right) dt \\ &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}\left(1+\frac{1}{\beta^2\sigma^2}\right)}\, dt \\ &= \frac{1}{2} - \sqrt{\frac{\beta^2\sigma^2}{1+\beta^2\sigma^2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{s^2}{2}}\, ds \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{\beta^2\sigma^2}{1+\beta^2\sigma^2}}\right) \end{aligned}$$

where we have used the change of variables $s = t\sqrt{\frac{1+\beta^2\sigma^2}{\beta^2\sigma^2}}$.

2. This is similar to case 1 with $\beta = \sqrt{\frac{2E_b}{N_0}}$, therefore

$$E[P_b] = \frac{1}{2}\left(1 - \sqrt{\frac{\sigma^2 \frac{2E_b}{N_0}}{1 + \sigma^2 \frac{2E_b}{N_0}}}\right)$$

3. Obviously here

$$E[P_b] = \frac{1}{2}\left(1 - \sqrt{\frac{\sigma^2 \frac{E_b}{N_0}}{1 + \sigma^2 \frac{E_b}{N_0}}}\right)$$

4. Note that for $\sigma^2 \frac{E_b}{N_0} \gg 1$, we have

$$\begin{aligned} \sqrt{\frac{\sigma^2 \frac{2E_b}{N_0}}{1 + \sigma^2 \frac{2E_b}{N_0}}} &= \sqrt{1 - \frac{1}{1 + \sigma^2 \frac{2E_b}{N_0}}} \\ &\approx 1 - \frac{1}{2 + 4\sigma^2 \frac{E_b}{N_0}} \end{aligned}$$

where we have used the approximation that for small ϵ, $\sqrt{1-\epsilon} \approx 1 - \frac{\epsilon}{2}$. From the above we have

$$E[P_b] = \begin{cases} \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}}\right) \approx \frac{1}{4\,\text{SNR}} & \text{antipodal} \\ \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{2+\text{SNR}}}\right) \approx \frac{1}{2\,\text{SNR}} & \text{orthogonal} \end{cases}$$

5. We have

$$\begin{aligned} E\left[e^{-\beta X^2}\right] &= \int_0^\infty e^{-\beta x^2}\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\,dx \\ &= \int_0^\infty \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}(1+2\beta\sigma^2)}\,dx \\ &= \frac{1}{1+2\beta\sigma^2}\int_0^\infty te^{-\frac{t^2}{2}}\,dt \\ &= \frac{1}{1+2\beta\sigma^2} \\ &\approx \frac{1}{2\beta\sigma^2} \qquad \text{for } \beta\sigma^2 \gg 1 \end{aligned}$$

where we have used the change of variables $t = x\sqrt{\frac{1+2\beta\sigma^2}{\sigma^2}}$. We will see later that the error probability for noncoherent detection of BFSK is $P_b = \frac{1}{2}e^{-\frac{\mathcal{E}_b}{2N_0}}$ and for binary DPSK is $P_b = \frac{1}{2}e^{-\frac{\mathcal{E}_b}{N_0}}$. If we have Rayleigh attenuation as in part 2, we can substitute $\beta = \frac{\mathcal{E}_b}{2N_0}$ and $\beta = \frac{\mathcal{E}_b}{N_0}$ and obtain

$$E[P_b] = \begin{cases} \frac{1}{2+\overline{\text{SNR}}} \approx \frac{1}{\overline{\text{SNR}}} & \text{noncoherent BFSK} \\ \frac{1}{2+2\,\overline{\text{SNR}}} \approx \frac{1}{2\,\overline{\text{SNR}}} & \text{binary DPSK} \end{cases}$$

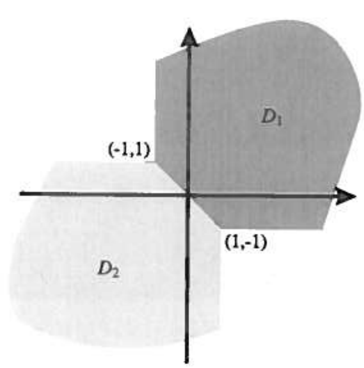
As noticed from parts 4 and 5, all error probabilities are inversely proportional to $\overline{\text{SNR}}$.

Problem 4.45

Here the noise is not Gaussian, therefore none of the results of Gaussian noise can be used. We start from the MAP rule

$$\begin{aligned} D_1 &= \{\boldsymbol{r} : p(\boldsymbol{s}_1|\boldsymbol{r}) > p(\boldsymbol{s}_2|\boldsymbol{r})\} \\ &= \{\boldsymbol{r} : p(\boldsymbol{s}_1)p(\boldsymbol{r}|\boldsymbol{s}_1) > p(\boldsymbol{s}_2)p(\boldsymbol{r}|\boldsymbol{s}_2)\} \\ &= \{\boldsymbol{r} : p(\boldsymbol{r}|\boldsymbol{s}_1) > p(\boldsymbol{r}|\boldsymbol{s}_2)\} \qquad \text{since the signals are equiprobable} \\ &= \{\boldsymbol{r} : \frac{1}{4}e^{-|r_1-1|-|r_2-1|} > \frac{1}{4}e^{-|r_1+1|-|r_2+1|}\} \quad \text{since noise components are independent} \\ &= \{\boldsymbol{r} : |r_1+1|+|r_2+1| > |r_1-1|+|r_2-1|\} \end{aligned}$$

If $r_1, r_2 > 1$, then $D_1 = \{\boldsymbol{r} : r_1+1+r_2+1 > r_1-1+r_2-1\}$ which is always satisfied. Therefore the entire $r_1, r_2 > 1$ region belongs to D_1. Similarly it can be shown that the entire $r_1, r_2 < -1$ region belongs to D_2. For $r_1 > 1$ and $r_2 < -1$ we have $D_1 = \{\boldsymbol{r} : r_1+1-r_2-1 > r_1-1-r_2+1\}$ or $0 > 0$, i.e.,$r_1, r_2 < -1$ can be either in D_1 or D_2. Similarly we can consider other regions of the plane. The final result is shown in the figure below. Regions D_1 and D_2 are shown in the figure and the rest of the plane can be either D_1 or D_2.



Problem 4.59

1. For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p+1-p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned}
P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\
&\stackrel{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\
&= (1-p-p)^n = (1-2p)^n
\end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned}
P_{\text{even}} + P_{\text{odd}} &= 1 \\
P_{\text{even}} - P_{\text{odd}} &= (1-2p)^n
\end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1-(1-2p)^n)$$

2. Expanding the quantity $(1-2p)^n$, we obtain

$$(1-2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1-1+n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

Problem 4.60

The overall probability of error is approximated by (see 4-10-2)

$$P(e) = KQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.61$$

Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2/2 = 15.7$

Problem 4.61

1. The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad D = 3\times 0.3048 \text{ m}$$

we obtain :

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

2. The effective radiated power is :

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

3. The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995\times 10^{-9} = -85.23 \text{ dB } = -55.23 \text{ dBm}$$

Note that :

$$\text{dBm } = 10\log_{10}\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log_{10}(\text{power in Watts })$$

Problem 4.62

1. The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \qquad \text{and} \qquad D = 1 \text{ m}$$

we obtain :

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

2. The effective radiated power is :

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

3. The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB} = -67.20 \text{ dBm}$$

Problem 4.63

The wavelength of the transmitted signal is:

$$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}$$

The gain of the parabolic antenna is:

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.6 \left(\frac{\pi 10}{0.03}\right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is:

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4 \times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

Problem 4.64

1. Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by :

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.5 \left(\frac{3.14159 \times 50}{\frac{3\times 10^8}{2\times 10^9}}\right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

2. If $\frac{\mathcal{E}_b}{N_0} = 10$ dB $= 10$, then

$$R = \frac{P_R}{N_0}\left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

Problem 4.65

The wavelength of the transmission is :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain :

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \Longrightarrow 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation (see 5-5-6) :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} \Longrightarrow P_T = \frac{P_R}{G_T G_R}\left(4\pi \frac{d}{\lambda}\right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

Problem 5.16

The PDF of the carrier phase error ϕ_e, is given by :

$$p(\phi_e) = \frac{1}{\sqrt{2\pi}\sigma_\phi} e^{-\frac{\phi_e^2}{2\sigma_\phi^2}}$$

Thus the average probability of error is :

$$\begin{aligned}
\bar{P}_2 &= \int_{-\infty}^{\infty} P_2(\phi_e) p(\phi_e) d\phi_e \\
&= \int_{-\infty}^{\infty} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}\cos^2\phi_e}\right] p(\phi_e) d\phi_e \\
&= \frac{1}{2\pi\sigma_\phi} \int_{-\infty}^{\infty} \int_{\sqrt{\frac{2\mathcal{E}_b}{N_0}\cos^2\phi_e}}^{\infty} \exp\left[-\frac{1}{2}\left(x^2 + \frac{\phi_e^2}{\sigma_\phi^2}\right)\right] dx d\phi_e
\end{aligned}$$

Problem 7.13

a. Interchanging the first and third rows, we obtain the systematic form :

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

b.

$$\mathbf{H} = [\mathbf{P}^T | \mathbf{I}_4] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c. Since we have a (7,3) code, there are $2^3 = 8$ valid codewords, and 2^4 possible syndromes. From these syndromes the all-zero one corresponds to no error, 7 will correspond to single errors and 8 will correspond to double errors (the choice is not unique) :

Error pattern	Syndrome
0 0 0 0 0 0 0	0 0 0 0
0 0 0 0 0 0 1	0 0 0 1
0 0 0 0 0 1 0	0 0 1 0
0 0 0 0 1 0 0	0 1 0 0
0 0 0 1 0 0 0	1 0 0 0
0 0 1 0 0 0 0	1 1 0 1
0 1 0 0 0 0 0	0 1 1 1
1 0 0 0 0 0 0	1 1 1 0
1 0 0 0 0 0 1	1 1 1 1
1 0 0 0 0 1 0	1 1 0 0
1 0 0 0 1 0 0	1 0 1 0
1 0 0 1 0 0 0	0 1 1 0
1 0 1 0 0 0 0	0 0 1 1
1 1 0 0 0 0 0	1 0 0 1
0 1 0 0 0 1 0	0 1 0 1
0 0 0 1 1 0 1	1 0 1 1

d. We note that there are 3 linearly independent columns in $\mathbf{H}$, hence there is a codeword $\mathbf{C}_m$ with weight w_m =4 such that $\mathbf{C}_m\mathbf{H}^T = 0$. Accordingly : $d_{\min} = 4$. This can be also obtained by generating all 8 codewords for this code and checking their minimum weight.

e. 101 generates the codeword : $101 \rightarrow \mathbf{C} = 1010011$. Then : $\mathbf{C}\mathbf{H}^T = [0000]$.

Problem 7.14

We have $(n, k) = (n, n-k)$, thus $n - k = k$, hence $n = 2k$ is even and $R = k/n = 1/2$.

Problem 7.15

It is easily verified that each codeword is orthogonal to itself and to all other codewords, hence the code is self dual.

Problem 7.16

$$\mathbf{G}_a = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{G}_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Message $\mathbf{X_m}$	$\mathbf{C_{ma}} = \mathbf{X_m G_a}$	$\mathbf{C_{mb}} = \mathbf{X_m G_b}$
0 0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 1 1	0 0 0 1 0 1 1
0 0 1 0	0 0 1 0 1 1 0	0 0 1 0 1 1 0
0 0 1 1	0 0 1 1 1 0 1	0 0 1 1 1 0 1
0 1 0 0	0 1 0 1 1 0 0	0 1 0 0 1 1 1
0 1 0 1	0 1 0 0 1 1 1	0 1 0 1 1 0 0
0 1 1 0	0 1 1 1 0 1 0	0 1 1 0 0 0 1
0 1 1 1	0 1 1 0 0 0 1	0 1 1 1 0 1 0
1 0 0 0	1 0 1 1 0 0 0	1 0 0 0 1 0 1
1 0 0 1	1 0 1 0 0 1 1	1 0 0 1 1 1 0
1 0 1 0	1 0 0 1 1 1 0	1 0 1 0 0 1 1
1 0 1 1	1 0 0 0 1 0 1	1 0 1 1 0 0 0
1 1 0 0	1 1 1 0 1 0 0	1 1 0 0 0 1 0
1 1 0 1	1 1 1 1 1 1 1	1 1 0 1 0 0 1
1 1 1 0	1 1 0 0 0 1 0	1 1 1 0 1 0 0
1 1 1 1	1 1 0 1 0 0 1	1 1 1 1 1 1 1

As we see, the two generator matrices generate the same set of codewords.

Problem 7.17

The weight distribution of the (7,4) Hamming code is $(n = 7)$:

$$\begin{aligned} A(x) &= \tfrac{1}{8}\left[(1+x)^7 + 7(1+x)^3(1-x)^4\right] \\ &= \tfrac{1}{8}\left[8 + 56x^3 + 56x^4 + 8x^7\right] \\ &= 1 + 7x^3 + 7x^4 + x^7 \end{aligned}$$

Hence, we have 1 codeword of weight zero, 7 codewords of weight 3, 7 codewords of weight 4, and one codeword of weight 7. which agrees with the codewords given in Table 7-9-2.

Problem 7.25

The number of errors is d and the number of components received with no error is $n-d$. Therefore,

$$P(y|x) = p^d(1-p)^{n-d} = (1-p)^n \left(\frac{p}{1-p}\right)^d$$

If $p < \frac{1}{2}$, then $p/(1-p) < 1$ and $P(y|x)$ is a decreasing function of d, hence an ML decoder that maximizes $P(y|x)$ should minimize d. If $p > \frac{1}{2}$, then an ML decoder should maximize d.

Problem 7.43

We can determine $\mathbf{G}$, in a systematic form, from the generator polynomial $g(p) = p^3 + p^2 + 1$:

$$\begin{array}{l} p^6 = (p^3+p^2+p)g(p) + p^2 + p \\ p^5 = (p^2+p+1)g(p) + p + 1 \\ p^4 = (p+1)g(p) + p^2 + p + 1 \\ p^3 = g(p) + p^2 + 1 \end{array} \quad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the parity check matrix for the extended code will be (according to 7-8-5) :

$$\mathbf{H}_e = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and in systematic form (we add rows 1,2,3 to the last one) :

$$\mathbf{H}_{es} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{G}_{es} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Note that $\mathbf{G}_{es}$ can be obtained from the generator matrix $\mathbf{G}$ for the initial code, by adding an overall parity check bit. The code words for the extended systematic code are :

Message $\mathbf{X_m}$	Codeword $\mathbf{C_m}$
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 1 0 1 1
0 0 1 0	0 0 1 0 1 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 0 1 1 1
0 1 0 1	0 1 0 1 1 1 0 0
0 1 1 0	0 1 1 0 1 0 0 1
0 1 1 1	0 1 1 1 0 0 1 0
1 0 0 0	1 0 0 0 1 1 0 1
1 0 0 1	1 0 0 1 0 1 1 0
1 0 1 0	1 0 1 0 0 0 1 1
1 0 1 1	1 0 1 1 1 0 0 0
1 1 0 0	1 1 0 0 1 0 1 0
1 1 0 1	1 1 0 1 0 0 0 1
1 1 1 0	1 1 1 0 0 1 0 0
1 1 1 1	1 1 1 1 1 1 1 1

An alternative way to obtain the codewords for the extended code is to add an additional check bit to the codewords of the initial (7,4) code which are given in Table 8-1-2. As we see, the minimum weight is 4 and hence : $d_{\min} = 4$.

Problem 7.44

a. We have obtained the generator matrix $\mathbf{G}$ for the (15,11) Hamming code in the solution of Problem 8.4. The shortened code will have a generator matrix $\mathbf{G}_s$ obtained by $\mathbf{G}$, by dropping its first 7 rows and the first 7 columns or :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

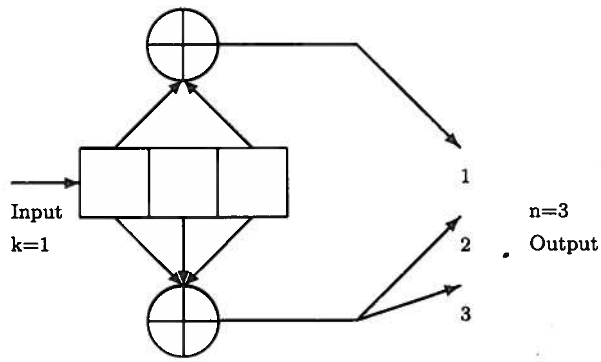
Then the possible messages and the codewords corresponding to them will be :

Message $\mathbf{X}_m$	Codeword $\mathbf{C}_m$
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 0 1 1
0 0 1 0	0 0 1 0 0 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 1 1 0 0
0 1 0 1	0 1 0 1 1 1 1 1
0 1 1 0	0 1 1 0 1 0 1 0
0 1 1 1	0 1 1 1 1 0 0 1
1 0 0 0	1 0 0 0 1 0 1 1
1 0 0 1	1 0 0 1 1 0 0 0
1 0 1 0	1 0 1 0 1 1 0 1
1 0 1 1	1 0 1 1 1 0 1 0
1 1 0 0	1 1 0 0 0 1 1 1
1 1 0 1	1 1 0 1 0 1 0 0
1 1 1 0	1 1 1 0 0 0 0 1
1 1 1 1	1 1 1 1 0 0 1 0

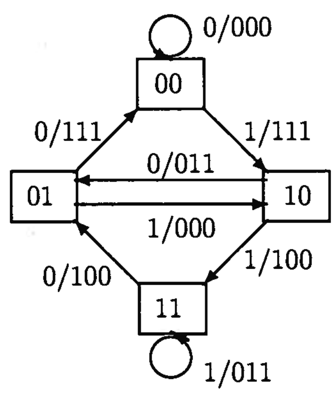
b. As we see the minimum weight and hence the minimum distance is 3 : $d_{\min} = 3$.

Problem 8.1

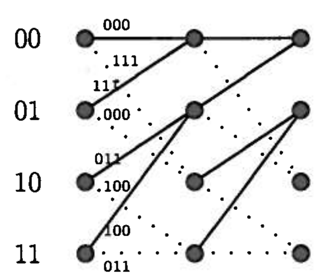
(a) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



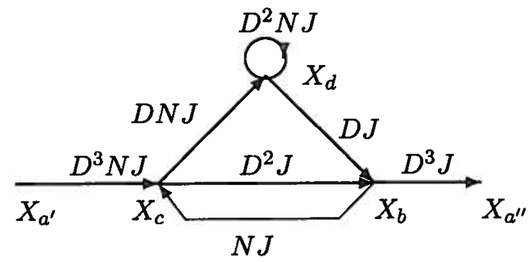
(b) The state transition diagram for this code is depicted in the next figure.



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + NJX_b \\ X_b &= D^2JX_c + DJX_d \\ X_d &= DNJX_c + D^2NJX_d \\ X_{a''} &= D^3JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^8NJ^3(1+NJ-D^2NJ)}{1-D^2NJ(1+NJ^2+J-D^2J^2)}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

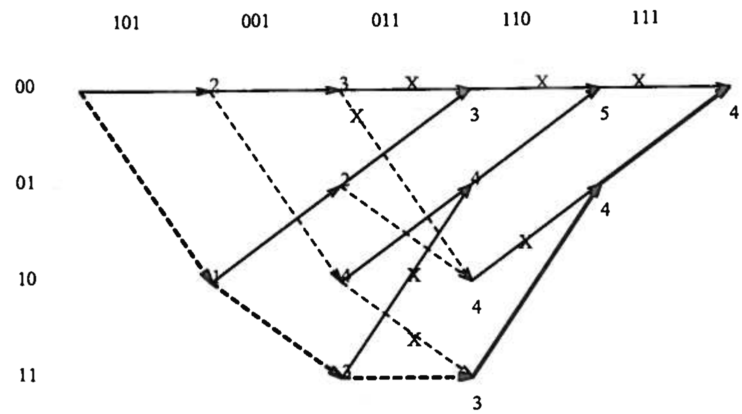
$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^8(1-2D^2)}{1-D^2(3-D^2)} = D^8 + 2D^{10} + \cdots$$

Hence, $d_{\text{free}} = 8$

(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.2

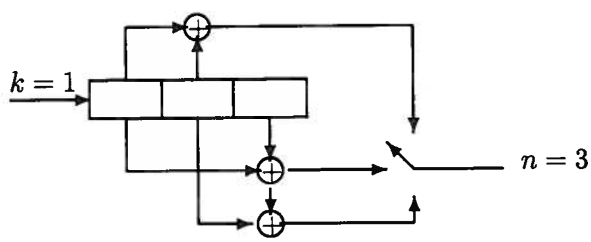
The code of Problem 8-1 is a $(3,1)$ convolutional code with $K = 3$. The length of the received sequence $\mathbf{y}$ is 15. This means that 5 symbols have been transmitted, and since we assume that the information sequence has been padded by two 0's, the actual length of the information sequence is 3. The following figure depicts 5 frames of the trellis used by the Viterbi decoder. The numbers on the nodes denote the metric (Hamming distance) of the survivor paths (the non-survivor paths are shown with an X). In the case of a tie of two merging paths at a node, we have purged the upper path.



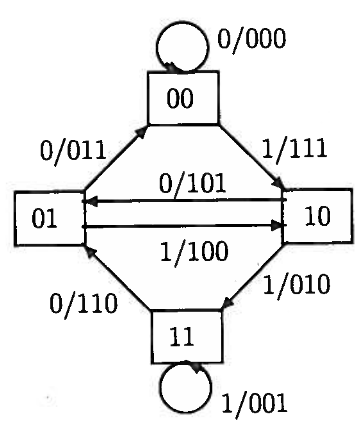
The decoded sequence is $\{111, 100, 011, 100, 111\}$ (i.e the path with the minimum final metric - heavy line) and corresponds to the information sequence $\{1, 1, 1\}$ followed by two zeros.

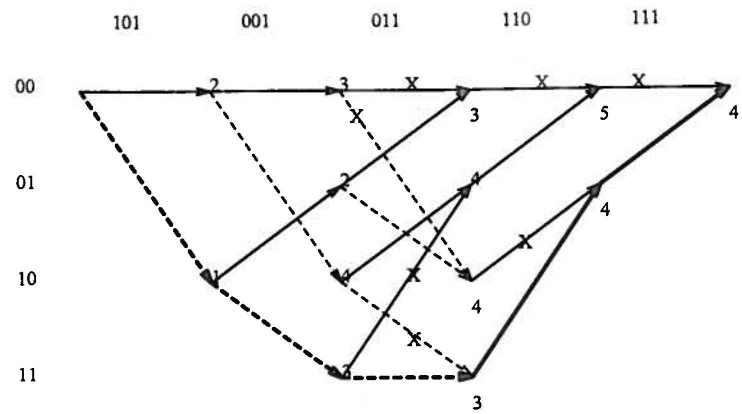
Problem 8.3

(a) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



(b) The state transition diagram for this code is shown below

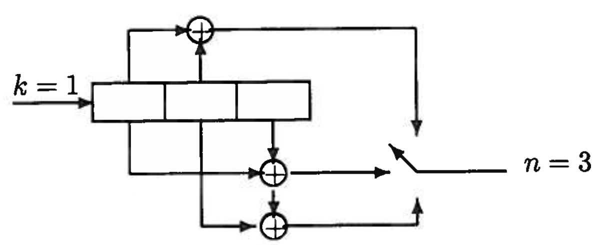




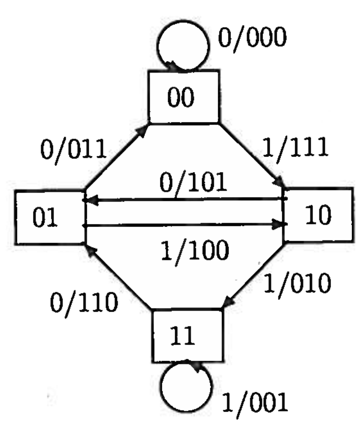
The decoded sequence is $\{111, 100, 011, 100, 111\}$ (i.e the path with the minimum final metric - heavy line) and corresponds to the information sequence $\{1, 1, 1\}$ followed by two zeros.

Problem 8.3

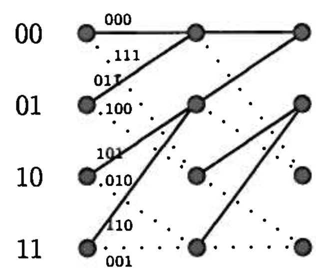
(a) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



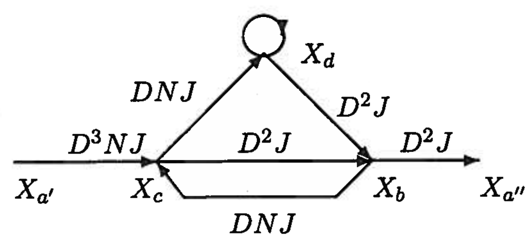
(b) The state transition diagram for this code is shown below



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$
\begin{aligned}
X_c &= D^3NJX_{a'} + DNJX_b \\
X_b &= D^2JX_c + D^2JX_d \\
X_d &= DNJX_c + DNJX_d \\
X_{a''} &= D^2JX_b
\end{aligned}
$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

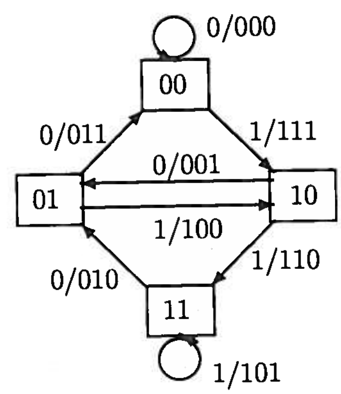
$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

Hence, $d_{\text{free}} = 7$

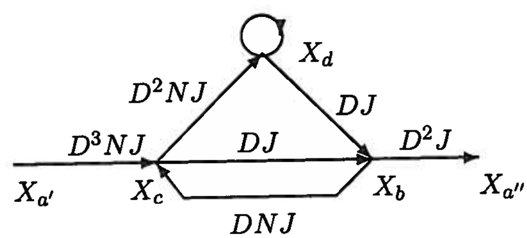
(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.4

(a) The state transition diagram for this code is depicted in the next figure.



(b) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned}
X_c &= D^3NJX_{a'} + DNJX_b \\
X_b &= DJX_c + DJX_d \\
X_d &= D^2NJX_c + D^2NJX_d \\
X_{a''} &= D^2JX_b
\end{aligned}$$

Eliminating X_b, X_c and X_d results in

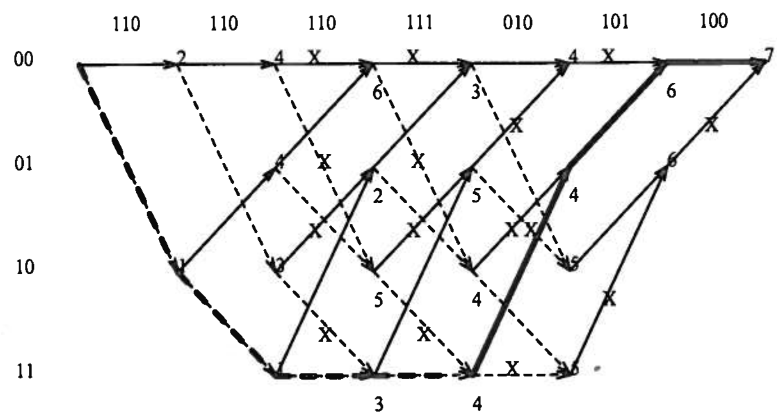
$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6NJ^3}{1 - D^2NJ - D^2NJ^2}$$

(c) To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^{10} + \cdots$$

Hence, $d_{\text{free}} = 6$

(d) The following figure shows 7 frames of the trellis diagram used by the Viterbi decoder. It is assumed that the input sequence is padded by two zeros, so that the actual length of the information sequence is 5. The numbers on the nodes indicate the Hamming distance of the survivor paths. The deleted branches have been marked with an X. In the case of a tie we deleted the upper branch. The survivor path at the end of the decoding is denoted by a thick line.



The information sequence is 11110 and the corresponding codeword 111 110 101 101 010 011 000...

(e) An upper to the bit error probability of the code is given by

$$P_b \leq \left. \frac{dT(D,N,J=1)}{dN} \right|_{N=1, D=\sqrt{4p(1-p)}}$$

But

$$\frac{dT(D,N,1)}{dN} = \frac{d}{dN}\left[\frac{D^6 N}{1-2D^2 N}\right] = \frac{D^6 - 2D^8(1-N)}{(1-2D^2N)^2}$$

and since $p = 10^{-5}$, we obtain

$$P_b \leq \left. \frac{D^6}{(1-2D^2)^2} \right|_{D=\sqrt{4p(1-p)}} \approx 6.14 \cdot 10^{-14}$$

Problem 8.17

The encoder is shown in Probl. 8.8. The channel is binary symmetric and the metric for Viterbi decoding is the Hamming distance. The trellis and the surviving paths are illustrăted in the following figure :