P 1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $p(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned} p(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\ p(X > 4 \times 10^{-4}) &= Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5} \\ p(-2 \times 10^{-4} < X \leq 10^{-4}) &= 1 - Q(1) - Q(2) = .8182 \end{aligned}$$

2)

$$p(X > 10^{-4} | X > 0) = \frac{p(X > 10^{-4},\ X > 0)}{p(X > 0)} = \frac{p(X > 10^{-4})}{p(X > 0)} = \frac{.159}{.5} = .318$$

3) $y = g(x) = xu(x)$. Clearly $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$. If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$. Hence, $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$. $F_Y(y)$ is discontinuous at $y = 0$ and the jump of the discontinuity equals $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2}$$

In summary the PDF $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y)$$

The general expression for finding $f_Y(y)$ can not be used because $g(x)$ is constant for some interval so that there is an uncountable number of solutions for x in this interval.

4)

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} y \left[f_X(y)u(y) + \frac{1}{2}\delta(y) \right] dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma}{\sqrt{2\pi}} \end{aligned}$$

5) $y = g(x) = |x|$. For a given $y > 0$ there are two solutions to the equation $y = g(x) = |x|$, that is $x_{1,2} = \pm y$. Hence for $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|\text{sgn}(x_1)|} + \frac{f_X(x_2)}{|\text{sgn}(x_2)|} = f_X(y) + f_X(-y) \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \end{aligned}$$

For $y < 0$ there are no solutions to the equation $y = |x|$ and $f_Y(y) = 0$.

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{2\sigma}{\sqrt{2\pi}}$$

P2)

$$\begin{aligned} E[Y] &= \int_0^\infty y f_Y(y)dy \geq \int_\alpha^\infty y f_Y(y)dy \\ &\geq \alpha \int_\alpha^\infty y f_Y(y)dy = \alpha p(Y \geq \alpha) \end{aligned}$$

Thus $p(Y \geq \alpha) \leq E[Y]/\alpha$.

2) Clearly $p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2)$. Thus using the results of the previous question we obtain

$$p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2) \leq \frac{E[(X - E[X])^2]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

P3) The characteristic function of the binomial distribution is

$$\begin{aligned} \psi_X(v) &= \sum_{k=0}^{n} e^{jvk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^{jv})^k (1-p)^{n-k} = (pe^{jv} + (1-p))^n \end{aligned}$$

Thus

$$\begin{aligned} E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}(pe^{jv} + (1-p))^n \bigg|_{v=0} = \frac{1}{j} n(pe^{jv} + (1-p))^{n-1} pje^{jv} \bigg|_{v=0} \\ &= n(p + 1 - p)^{n-1} p = np \\ E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}(pe^{jv} + (1-p))^n \bigg|_{v=0} \\ &= (-1)\frac{d}{dv}\left[n(pe^{jv} + (1-p))^{n-1} pje^{jv}\right]\bigg|_{v=0} \\ &= \left[n(n-1)(pe^{jv} + (1-p))^{n-2} p^2 e^{2jv} + n(pe^{jv} + (1-p))^{n-1} pe^{jv}\right]\bigg|_{v=0} \\ &= n(n-1)(p + 1 - p)p^2 + n(p + 1 - p)p \\ &= n(n-1)p^2 + np \end{aligned}$$

Hence the variance of the binomial distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

P4) 1) $f_{X,Y}(x, y)$ is a PDF, hence its integral over the supporting region of x, and y is 1.

$$\begin{aligned} \int_0^\infty \int_y^\infty f_{X,Y}(x, y)dxdy &= \int_0^\infty \int_y^\infty Ke^{-x-y}dxdy \\ &= K\int_0^\infty e^{-y} \int_y^\infty e^{-x}dxdy \\ &= K\int_0^\infty e^{-2y}dy = K(-\frac{1}{2})e^{-2y}\bigg|_0^\infty = K\frac{1}{2} \end{aligned}$$

Thus K should be equal to 2.

2)

$$\begin{aligned} f_X(x) &= \int_0^x 2e^{-x-y}dy = 2e^{-x}(-e^{-y})\Big|_0^x = 2e^{-x}(1-e^{-x}) \\ f_Y(y) &= \int_y^\infty 2e^{-x-y}dy = 2e^{-y}(-e^{-x})\Big|_y^\infty = 2e^{-2y} \end{aligned}$$

3)

$$\begin{aligned} f_X(x)f_Y(y) &= 2e^{-x}(1-e^{-x})2e^{-2y} = 2e^{-x-y}2e^{-y}(1-e^{-x}) \\ &\neq 2e^{-x-y} = f_{X,Y}(x,y) \end{aligned}$$

Thus X and Y are not independent.

4) If $x < y$ then $f_{X|Y}(x|y) = 0$. If $x \geq y$, then with $u = x - y \geq 0$ we obtain

$$f_U(u) = f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y} = e^{-u}$$

5)

$$\begin{aligned} E[X|Y=y] &= \int_y^\infty xe^{-x+y}dx = e^y \int_y^\infty xe^{-x}dx \\ &= e^y \left[-xe^{-x}\Big|_y^\infty + \int_y^\infty e^{-x}dx \right] \\ &= e^y(ye^{-y} + e^{-y}) = y + 1 \end{aligned}$$

6) In this part of the problem we will use extensively the following definite integral

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx = \frac{1}{\mu^\nu}(\nu-1)!$$

$$\begin{aligned} E[XY] &= \int_0^\infty \int_y^\infty xy2e^{-x-y}dxdy = \int_0^\infty 2ye^{-y} \int_y^\infty xe^{-x}dxdy \\ &= \int_0^\infty 2ye^{-y}(ye^{-y}+e^{-y})dy = 2\int_0^\infty y^2e^{-2y}dy + 2\int_0^\infty ye^{-2y}dy \\ &= 2\frac{1}{2^3}2! + 2\frac{1}{2^2}1! = 1 \end{aligned}$$

$$\begin{aligned} E[X] &= 2\int_0^\infty xe^{-x}(1-e^{-x})dx = 2\int_0^\infty xe^{-x}dx - 2\int_0^\infty xe^{-2x}dx \\ &= 2 - 2\frac{1}{2^2} = \frac{3}{2} \\ E[Y] &= 2\int_0^\infty ye^{-2y}dy = 2\frac{1}{2^2} = \frac{1}{2} \\ E[X^2] &= 2\int_0^\infty x^2e^{-x}(1-e^{-x})dx = 2\int_0^\infty x^2e^{-x}dx - 2\int_0^\infty x^2e^{-2x}dx \\ &= 2\cdot 2! - 2\frac{1}{2^3}2! = \frac{7}{2} \\ E[Y^2] &= 2\int_0^\infty y^2e^{-2y}dy = 2\frac{1}{2^3}2! = \frac{1}{2} \end{aligned}$$

Hence,

$$COV(X,Y) = E[XY] - E[X]E[Y] = 1 - \frac{3}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

and

$$\rho_{X,Y} = \frac{COV(X,Y)}{(E[X^2]-(E[X])^2)^{1/2}(E[Y^2]-(E[Y])^2)^{1/2}} = \frac{1}{\sqrt{5}}$$

P5)

The binormal joint density function is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\times\right. \\ &\left.\left[\frac{(x-m_1)^2}{\sigma_1^2}+\frac{(y-m_2)^2}{\sigma_2^2}-\frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2}\right]\right\} \\ &= \frac{1}{\sqrt{(2\pi)^n\det(C)}}\exp\left\{-(\mathbf{z}-\mathbf{m})C^{-1}(\mathbf{z}-\mathbf{m})^t\right\} \end{aligned}$$

where $\mathbf{z} = [x \;\; y]$, $\mathbf{m} = [m_1 \;\; m_2]$ and

$$C = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

1) With

$$C = \begin{pmatrix} 4 & -4 \\ -4 & 9 \end{pmatrix}$$

we obtain $\sigma_1^2 = 4$, $\sigma_2^2 = 9$ and $\rho\sigma_1\sigma_2 = -4$. Thus $\rho = -\frac{2}{3}$.

2) The transformation $Z = 2X + Y$, $W = X - 2Y$ is written in matrix notation as

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$$

The ditribution $f_{Z,W}(z,w)$ is binormal with mean $\mathbf{m}' = \mathbf{m}A^t$, and covariance matrix $C' = ACA^t$. Hence

$$C' = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & -4 \\ -4 & 9 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 2 & 56 \end{pmatrix}$$

The off-diagonal elements of C' are equal to $\rho\sigma_Z\sigma_W = COV(Z,W)$. Thus $COV(Z,W) = 2$.

3) Z will be Gaussian with variance $\sigma_Z^2 = 9$ and mean

$$m_Z = [\, m_1 \;\; m_2 \,]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

P6)

1) $f_{X,Y}(x,y)$ is a PDF and its integral over the supporting region of x and y should be one.

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{0}\int_{-\infty}^{0}\frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy + \int_{0}^{\infty}\int_{0}^{\infty}\frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\ &= \frac{K}{\pi}\int_{-\infty}^{0}e^{-\frac{x^2}{2}}dx\int_{-\infty}^{0}e^{-\frac{y^2}{2}}dx + \frac{K}{\pi}\int_{0}^{\infty}e^{-\frac{x^2}{2}}dx\int_{0}^{\infty}e^{-\frac{y^2}{2}}dx \\ &= \frac{K}{\pi}\left[2\left(\frac{1}{2}\sqrt{2\pi}\right)^2\right] = K \end{aligned}$$

Thus $K = 1$

2) If $x < 0$ then

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{0} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{0} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2} \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

If $x > 0$ then

$$\begin{aligned} f_X(x) &= \int_{0}^{\infty} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{0}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2} \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

Thus for every x, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which implies that $f_X(x)$ is a zero-mean Gaussian random variable with variance 1. Since $f_{X,Y}(x,y)$ is symmetric to its arguments and the same is true for the region of integration we conclude that $f_Y(y)$ is a zero-mean Gaussian random variable of variance 1.

3) $f_{X,Y}(x,y)$ has not the same form as a binormal distribution. For $xy < 0$, $f_{X,Y}(x,y) = 0$ but a binormal distribution is strictly positive for every x, y.

4) The random variables X and Y are not independent for if $xy < 0$ then $f_X(x) f_Y(y) \neq 0$ whereas $f_{X,Y}(x,y) = 0$.

5)

$$\begin{aligned} E[XY] &= \frac{1}{\pi} \int_{-\infty}^{0} \int_{-\infty}^{0} XY e^{-\frac{x^2+y^2}{2}} dx dy + \frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy \\ &= \frac{1}{\pi} \int_{-\infty}^{0} X e^{-\frac{x^2}{2}} dx \int_{-\infty}^{0} Y e^{-\frac{y^2}{2}} dy + \frac{1}{\pi} \int_{0}^{\infty} X e^{-\frac{x^2}{2}} dx \int_{0}^{\infty} Y e^{-\frac{y^2}{2}} dy \\ &= \frac{1}{\pi}(-1)(-1) + \frac{1}{\pi} = \frac{2}{\pi} \end{aligned}$$

Thus the random variables X and Y are correlated since $E[XY] \neq 0$ and $E[X] = E[Y] = 0$, so that $E[XY] - E[X]E[Y] \neq 0$.

6) In general $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. If $y > 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x \geq 0 \end{cases}$$

If $y \leq 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x > 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x < 0 \end{cases}$$

Thus

$$f_{X|Y}(x,y) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} u(xy)$$

which is not a Gaussian distribution.

P7) 1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin ($\tau = 0$). We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$S(f) = 1.2\text{sinc}^2(f) - \text{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $S(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

P8) 1)

$$\begin{aligned} m_X(t) &= E[X(t)] = E[X\cos(2\pi f_0 t)] + E[Y\sin(2\pi f_0 t)] \\ &= E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) \\ &= 0 \end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

2)

$$\begin{aligned} R_X(t+\tau, t) &= E[(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau))) \\ &\quad (X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))] \\ &= E[X^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[XY\cos(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] + \\ &\quad E[YX\sin(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[Y^2\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] \\ &= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0 \tau)] + \end{aligned}$$

$$
\begin{aligned}
& \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\
= \;& \sigma^2 \cos(2\pi f_0\tau)
\end{aligned}
$$

where we have used the fact that $E[XY] = 0$. Thus the process is stationary for $R_X(t+\tau, t)$ depends only on τ.

3) Since the process is stationary $P_X = R_X(0) = \sigma^2$.

4) If $\sigma_X^2 \neq \sigma_Y^2$, then

$$
m_X(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0
$$

and

$$
\begin{aligned}
R_X(t+\tau, t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + \\
& \quad E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t) \\
&= \frac{\sigma_X^2}{2}[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)] + \\
& \quad \frac{\sigma_Y^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\
&= \frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0(2t+\tau) + \\
& \quad \frac{\sigma_X^2 + \sigma_Y^2}{2}\cos(2\pi f_0\tau)
\end{aligned}
$$

The process is not stationary for $R_X(t+\tau, t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$. The power spectral density of $X(t)$ is

$$
P_X = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0 2t) + \frac{\sigma_X^2 + \sigma_Y^2}{2}\right)dt = \infty
$$