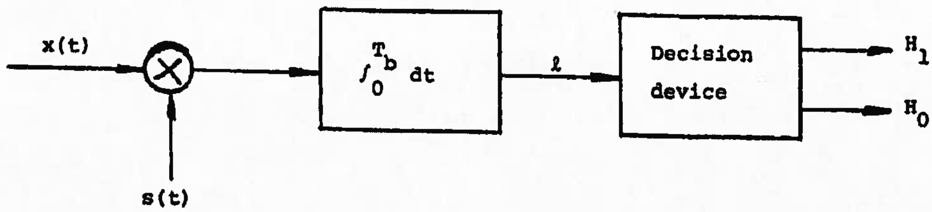
Solution to Assignment 3

Problem 1:

(a) ASK with coherent reception



Denoting the presence of symbol 1 or symbol 0 by hypothesis H_1 or H_0, respectively, we may write

H_1: $x(t) = s(t) + w(t)$

H_0: $x(t) = w(t)$

where $s(t) = A_c\cos(2\pi f_c t)$, with $A_c = \sqrt{2E_b/T_b}$. Therefore,

$$\ell = \int_0^{T_b} x(t)\, s(t)\, dt$$

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If $\ell > E_b/2$, the receiver decides in favor of symbol 1. If $\ell < E_b/2$, it decides in favor of symbol 0.

The conditional probability density functions of the random variable L, whose value is denoted by ℓ, are defined by

$$f_{L|0}(\ell|0) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp(-\frac{\ell^2}{N_0 E_b})$$

$$f_{L|1}(\ell|1) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp[-\frac{(\ell - E_b)^2}{N_0 E_b}]$$

The average probability of error is therefore,

$$P_e = p_0 \int_{E_b/2}^{\infty} f_{L|0}(\ell|0)d\ell + p_1 \int_{-\infty}^{E_b/2} f_{L|1}(\ell|1)d\ell$$

$$= \frac{1}{2} \int_{E_b/2}^{\infty} \frac{1}{\sqrt{\pi N_0 E_b}} \exp(-\frac{\ell^2}{N_0 E_b})d\ell + \frac{1}{2} \int_{-\infty}^{E_b/2} \frac{1}{\sqrt{\pi N_0 E_b}} \exp[-\frac{(\ell - E_b)^2}{N_0 E_b}]d\ell$$

$$= \frac{1}{\sqrt{\pi N_0 E_b}} \int_{E_b/2}^{\infty} \exp(-\frac{\ell^2}{N_0 E_b})d\ell$$

$$= \frac{1}{2} \operatorname{erfc}(\frac{1}{2}\sqrt{E_b/N_0})$$

(b) ASK with noncoherent reception

In this case, the signal s(t) is defined by

$$s(t) = A_c \cos(2\pi f_c t + \theta)$$

where $A_c = \sqrt{2 E_b/T_b}$, and

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

For the case when symbol 0 is transmitted, that is, under hypothesis H_0, we find that the random variable L, at the input of the decision device, is Rayleigh-distributed:

$$f_{L|0}(\ell|0) = \frac{4\ell}{N_0 T_b} \exp(-\frac{2\ell^2}{N_0 T_b})$$

For the case when symbol 1 is transmitted, that is, under hypothesis H_1, we find that the random variable L is Rician-distributed:

$$f_{L|1}(\ell|1) = \frac{4\ell}{N_0T_b} \exp\left(-\frac{\ell^2 + A_c^2T_b^2/4}{N_0T_b/2}\right) I_0\left(\frac{2\ell A_c}{N_0}\right)$$

where $I_0(2\ell A_c/N_0)$ is the modified Bessel function of the first kind of zero order.

Before we can obtain a solution for the error performance of the receiver, we have to determine a value for the threshold. Since symbols 1 and 0 occur with equal probability, the minimum probability of error criterion yields:

$$\exp\left(-\frac{A_c^2\, T_b}{2N_0}\right) I_0\left(\frac{2\ell A_c}{N_0}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad (1)$$

For large values of E_b/N_0, we may approximate $I_0(2\ell A_c/N_0)$ as follows:

$$I_0\left(\frac{2\ell A_c}{N_0}\right) \simeq \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}}$$

Using this approximation, we may rewrite Eq. (1) as follows:

$$\exp\left[\frac{A_c(4 - A_cT_b)}{2N_0}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{4\pi\ell A_c}{N_0}}$$

Taking the logarithm of both sides of this relation, we get

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{A_cT_b}{4} + \frac{1}{2}\sqrt{\frac{\pi\ell N_0}{A_c}}$$

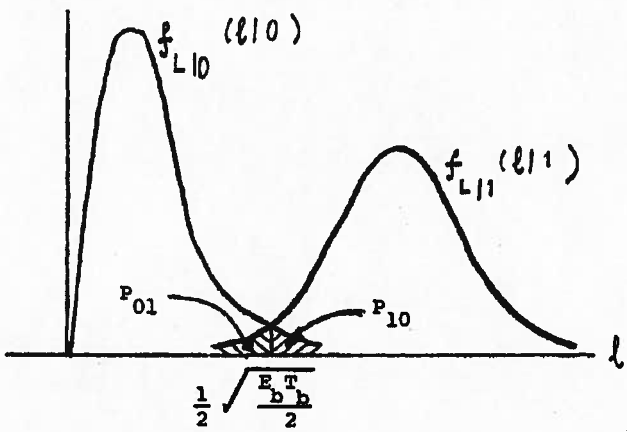
Neglecting the second term on the right hand side of this relation, and using the fact that

$$E_b = \frac{A_c^2T_b}{2}$$

we may write

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$$

The threshold $\frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$ is at the point corresponding to the crossover between the two probability density functions, as illustrated below.



The average probability of error is therefore

$$P_e = P_0\, P_{10} + P_1\, P_{01}$$

where

$$P_{10} = \int_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty} f_{L|0}(\ell|0)d\ell$$

$$= \int_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty} \frac{4\ell}{N_0T_b} \exp(-\frac{2\ell^2}{N_0T_b})d\ell$$

$$= \left[-\exp(-\frac{2\ell^2}{N_0T_b})\right]_{\sqrt{E_bT_b}/2\sqrt{2}}^{\infty}$$

$$= \exp(-\frac{E_b}{4N_0})$$

$$P_{01} = \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} f_{L|1}(\ell|1)d\ell$$

$$= \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} \frac{4\ell}{N_0T_b} \exp(-\frac{\ell^2 + A_c^2\, T_b^2/4}{N_0T_b/2})\; I_0(\frac{2\ell A_c}{N_0})d\ell$$

$$\approx \int_0^{\sqrt{E_bT_b}/2\sqrt{2}} \frac{4\ell}{N_0T_b} \exp(-\frac{\ell^2 + A_c^2\, T_b^2/4}{N_0T_b/2}) \cdot \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}}\; d\ell$$

$$= \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} \sqrt{\frac{2\ell}{A_c T_b}} \sqrt{\frac{2}{\pi N_0 T_b}} \exp\left[-\frac{(\ell - A_c T_b/2)^2}{N_0 T_b/2}\right] d\ell \tag{2}$$

The integrand in Eq. (2) is the product of $\sqrt{2\ell/A_c T_b}$ and the probability density function of a Gaussian random variable of mean $A_c T_b/2$ and variance $N_0 T_b/4$. For high values of E_b/N_0, the standard deviation $\sqrt{N_0 T_b/4}$ is much less than the threshold $\sqrt{E_b T_b}/2\sqrt{2}$. Consequently, the area under the portion of the curve from 0 to $\sqrt{E_b T_b}/2\sqrt{2}$ is quite small, that is, $P_{01} \simeq 0$. Then, we may approximate the average probability of error as

$$P_e \simeq p_o P_{10}$$

$$= \frac{1}{2} \exp\left(-\frac{E_b}{4N_0}\right)$$

where it is assumed that symbols 0 and 1 occur with equal probability.

Problem 2:

The transmitted binary PSK signal is defined by

$$s(t) = \begin{cases} \sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 1 \\ -\sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 0 \end{cases}$$

where the basis function $\phi(t)$ is defined by

$$\phi(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The locally generated basis function in the receiver is

$$\phi_{\text{rec}}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \varphi)$$

$$= \sqrt{\frac{2}{T_b}}[\cos(2\pi f_c t)\cos\varphi - \sin(2\pi f_c t)\sin\varphi]$$

where φ is the phase error. The correlator output is given by

$$y = \int_0^{T_b} x(t)\varphi_{\text{rec}}(t)dt$$

where

$$x(t) = s_k(t) + w(t), \qquad k = 1, 2$$

Assuming that f_c is an integer multiple of $1/T_b$, and recognizing that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ over the interval $0 \le t \le T_b$, we get

$$y = \pm\sqrt{E_b}\cos\varphi + W$$

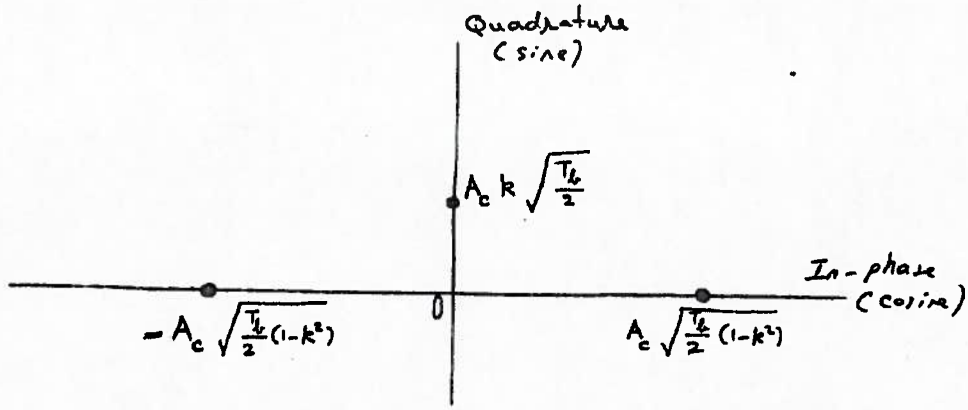
when the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0, and W is a zero-mean Gaussian variable of variance $N_0/2$. Accordingly, the average probability of error of the binary PSK system with phase error φ is given by

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b \cos\varphi}{N_0}}\right)$$

When $\varphi = 0$, this formula reduces to that for the standard PSK system equipped with perfect phase recovery. At the other extreme, when $\varphi = \pm 90°$, P_e attains its worst value of unity.

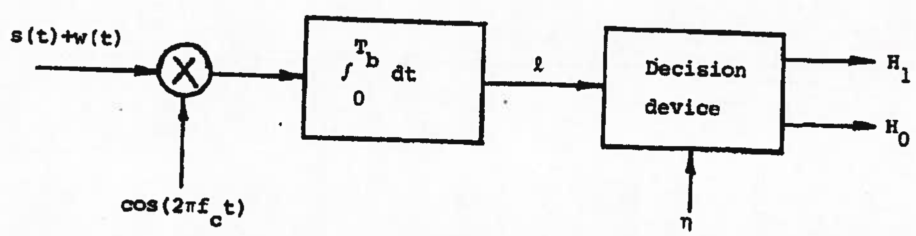
Problem 3:

(a) The signal-space diagram of the scheme described in this problem is two-dimensional, as shown by



This signal-space diagram differs from that of the conventional PSK signaling scheme in that it is two-dimensional, with a new signal point on the quadrature axis at $A_c k\sqrt{T_b/2}$. If k is reduced to zero, the above diagram reduces to the same form as that shown in Fig. 8.14.

(b)



The signal at the decision device input is

$$\ell = \pm \frac{A_c}{2}\sqrt{1-k^2}\,T_b + \int_0^{T_b} w(t)\cos(2\pi f_c t)dt \qquad (1)$$

Therefore, following a procedure similar to that used for evaluating the average probability of error for a conventional PSK system, we find that for the system defined by Eq. (1) the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-k^2)/N_0}\right)$$

where $E_b = \frac{1}{2} A_c^2 T_b$.

(c) For the case when $P_e = 10^{-4}$ and $k^2 = 0.1$, we get

$$10^{-4} = \frac{1}{2}\,\mathrm{erfc}(u)$$

where $u^2 = \dfrac{0.9\, E_b}{N_0}$

Using the approximation

$$\mathrm{erfc}(u) \simeq \frac{\exp(-u^2)}{\sqrt{\pi}\, u}$$

we obtain

$$\exp(-u^2) - 2\sqrt{\pi} \times 10^{-4}\, u = 0$$

The solution to this equation is $u = 2.64$. The corresponding value of E_b/N_0 is

$$\frac{E_b}{N_0} = \frac{(2.64)^2}{0.9} = 7.74$$

Expressed in decibels, this value corresponds to 8.9 dB.

(d) For a conventional PSK system, we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

In this case, we find that

$$\frac{E_b}{N_0} = (2.64)^2 = 6.92$$

Expressed in decibels, this value corresponds to 8.4 dB. Thus, the conventional PSK system requires 0.5 dB less in E_b/N_0 then the modified scheme described herein.

Problem 4:

The transmission bandwidth of 256-QAM signal is

$$B = \frac{2R_b}{\log_2 M}$$

where R_b is the bit rate given by $1/T_b$ and $M = 256$. Thus

$$B_{256} = \frac{2(1/T_b)}{\log_2 256} = \frac{2}{16T_b} = \frac{1}{8T_b}$$

The transmission bandwidth of 64-QAM is

$$B_{64} = \frac{2(1/T_b)}{\log_2 64} = \frac{2}{8T_b} = \frac{1}{4T_b}$$

Hence, the bandwidth advantage of 256-QAM over 64-QAM is

$$\frac{1}{4T_b} - \frac{1}{8T_b} = \frac{1}{8T_b}$$

The average energy of 256-QAM signal is

$$E_{256} = \frac{2(M-1)E_0}{3} = \frac{2(256-1)E_0}{3}$$

$$= 170E_0$$

where E_0 is the energy of the signal with the lowest amplitude. For the 64-QAM signal, we have

$$E_{64} = \frac{2(63)}{3}E_0 = 42E_0$$

Therefore, the increase in average signal energy resulting from the use of 256-QAM over 64-QAM, expressed in dBs, is

$$10\log_{10}\left(\frac{170E_0}{42E_0}\right) = 10\log_{10}(4)$$

$$= 6 \text{ dB}$$

Problem 5:

The probability of symbol error for 16-QAM is given by

$$P_e = 2\left(1-\frac{1}{\sqrt{M}}\right)\text{erfc}\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = 2\left(1-\frac{1}{4}\right)\text{erfc}\left(\sqrt{\frac{3E_{av}}{30N_0}}\right)$$

Solving this equation for E_{av}/N_0,

$$\frac{E_{av}}{N_0} = 58 = 17.6\text{dB}$$

The probability of symbol error for 16-PSK is given by

$$P_e = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/16)\right)$$

Solving this equation for E/N_0, we get

$$\frac{E}{N_0} = 142 = 21.5\text{dB}$$

Hence, on the average, the 16-PSK demands 21.5 - 17.6 = 3.9 dB more symbol energy than the 16-QAM for $P_e = 10^{-3}$.

Thus the 16-QAM requires about 4 dB less in signal energy than the 16-PSK for a fixed N_0 and $P_e = 10^{-3}$, However, for this advantage of the 16-QAM over the 16-PSK to be realized, the channel must be linear.

Problem 6

The bit duration is

$$T_b = \frac{1}{2.5\times10^6 \text{Hz}} = 0.4\ \mu s$$

The signal energy per bit is

$$E_b = \frac{1}{2} A_c^2 T_b$$
$$= \frac{1}{2} (10^{-6})^2 \times 0.4 \times 10^{-6} = 2 \times 10^{-19} \text{ joules}$$

(a) Coherent Binary FSK

The average probability of error is

$$P_e = \frac{1}{2} \text{erfc}(\sqrt{E_b/2N_0})$$

$$= \frac{1}{2} \text{erfc}(\sqrt{2\times10^{-19}/4\times10^{-20}})$$

$$= \frac{1}{2} \text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) \simeq \frac{\exp(-u^2)}{\sqrt{\pi}\, u}$$

we obtain the result

$$P_e = \frac{1}{2} \frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \times 10^{-3}$$

(b) MSK

$$P_e = \text{erfc}(\sqrt{E_b/N_0})$$

$$= \text{erfc}(\sqrt{10})$$

$$= \frac{\exp(-10)}{\sqrt{10\pi}}$$

$$= 0.81 \times 10^{-5}$$

(c) Noncoherent Binary FSK .

$$P_e = \frac{1}{2} \exp(-\frac{E_b}{2N_0})$$

$$= \frac{1}{2} \exp(-5)$$

$$= 3.37 \times 10^{-3}$$

Problem 7:

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{[\int_0^{T_b} s_0^2(t)dt]^{1/2} \; [\int_0^{T_b} s_1^2(t)dt]^{1/2}}$$

$$= \frac{A_c^2 \int_0^{T_b} \cos[2\pi(f_c + \frac{1}{2}\Delta f)t] \cos[2\pi(f_c - \frac{1}{2}\Delta f)t] \, dt}{[\frac{1}{2} A_c^2 T_b]^{1/2} \; [\frac{1}{2} A_c^2 T_b]^{1/2}}$$

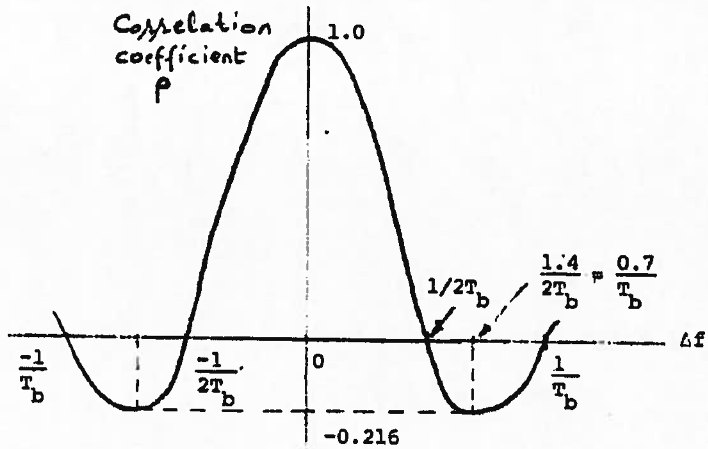
$$= \frac{1}{T_b} \int_0^{T_b} [\cos(2\pi\Delta f t) + \cos(4\pi f_c t)] \, dt$$

$$= \frac{1}{2\pi T_b} [\frac{\sin(2\pi\Delta f T_b)}{\Delta f} + \frac{\sin(4\pi f_c T_b)}{2f_c}] \qquad (1)$$

Since $f_c >> \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho = \frac{\sin(2\pi\Delta f T_b)}{2\pi T_b \; \Delta f} = \text{sinc}(2\Delta f T_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1.



Correlation
coefficient
ρ
1.0
1/2T_b
1.4/2T_b = 0.7/T_b
Δf
-1/T_b
-1/2T_b
0
1/T_b
-0.216

Fig. 1

$s_0(t)$ and $s_1(t)$ are othogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal, is $1/2T_b$.

(c) The average probability of error is given by

$$E_b = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-\rho)/2N_0}\right)$$

The most negative value of ρ is -0.216, occuring at $\Delta f = 0.7/T_b$. The minimum value of P_e is therefore

$$P_{e,min} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{0.608E_b/N_0}\right)$$

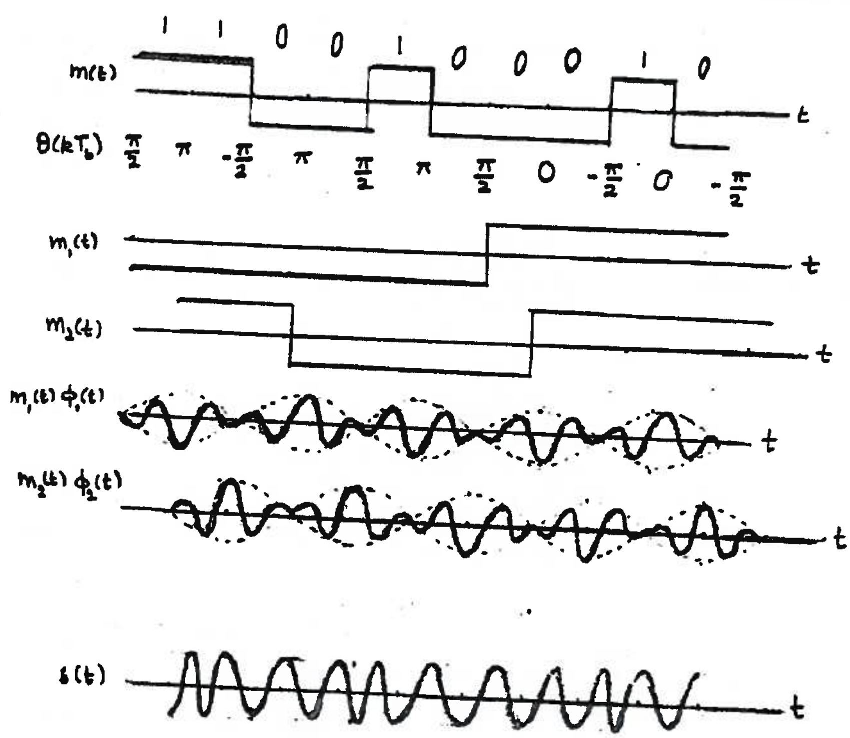
(d) For a coherent binary PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

Therefore, the E_b/N_0 of this coherent binary FSK system must be increased by the factor $1/0.608 = 1.645$ (or 2.16 dB) so as to realize the same average probability of error as a coherent binary PSK system.

Problem 8:

(a)



(b)