# SOLUTIONS TO ASSIGNMENT 4

Problem 1

(a) The Fourier transform of $h(t)$ is given by (using entry 5 of the Fourier-transform pairs Table A6.3)

$$H(f) = \frac{\sqrt{\pi}}{\alpha} \cdot \frac{1}{\sqrt{\pi}/\alpha} \exp\left(-\pi \times \frac{f^2}{\pi/\alpha^2}\right)$$

$$= \exp-(f^2\alpha^2) \qquad (1)$$

Substituting $\alpha = (\sqrt{\log 2/2}/W)$ into (1), we get

$$H(f) = \exp\left(-f^2 \ \frac{\log 2}{2} \times \frac{1}{W^2}\right)$$

$$= \exp\left(-\frac{\log 2}{2}\left(\frac{f}{W}\right)^2\right) \qquad (2)$$

Let $f_0$ denote the 3-dB cut-off frequency of the GMSK signal. Then, by definition,

$$|H(f_0)| = \frac{1}{\sqrt{2}}|H(0)|$$

$$= \frac{1}{\sqrt{2}}$$

Hence, from Eq. (2) it follows that

$$\exp\left(-\frac{\log 2}{2}\left(\frac{f_0}{W}\right)^2\right) = \frac{1}{\sqrt{2}}$$

or

$$\exp\left(\log 2\left(\frac{f_0}{W}\right)^2\right) = 2$$

Taking the logarithm of both sides, we readily find that

$f_0 = W$

The 3-dB bandwidth (cut-off frequency) of the filter used to shape GMSK signals is therefore $W$.

(b) The response of the filter to a rectangular pulse of unit amplitude and duration $T$ centered on the origin is given by

$$g(t) = \int_{-T/2}^{T/2} h(t-\tau)d\tau$$

$$= \int_{-T/2}^{T/2} \frac{\sqrt{\pi}}{\alpha}\exp\left[\frac{-\pi^2(t-\tau)^2}{\alpha^2}\right]d\tau \qquad (3)$$

Let $k = \dfrac{\pi(t-\tau)}{\alpha}$, and

$$dk = -\frac{\pi}{\alpha}d\tau$$

Hence, we may reqrite Eq. (3) as

$$g(t) = -\int_{k_1}^{k_2} \frac{\sqrt{\pi}}{\alpha}\exp(-k^2)\frac{\alpha}{\pi}dk \qquad (4)$$

where $k_1 = \dfrac{\pi(t+T/2)}{\alpha}$ and

$$k_2 = \frac{\pi(t-T/2)}{\alpha}$$

Equation (4) is finally rewritten as

$$g(t) = -\frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_0^{k_2}\exp(-k^2)dk + \frac{2}{\sqrt{\pi}}\int_{k_1}^{0}\exp(-k^2)dk\right\}$$

$$= -\frac{1}{2}\mathrm{erf}(k_2) + \frac{1}{2}\mathrm{erf}(k_1)$$

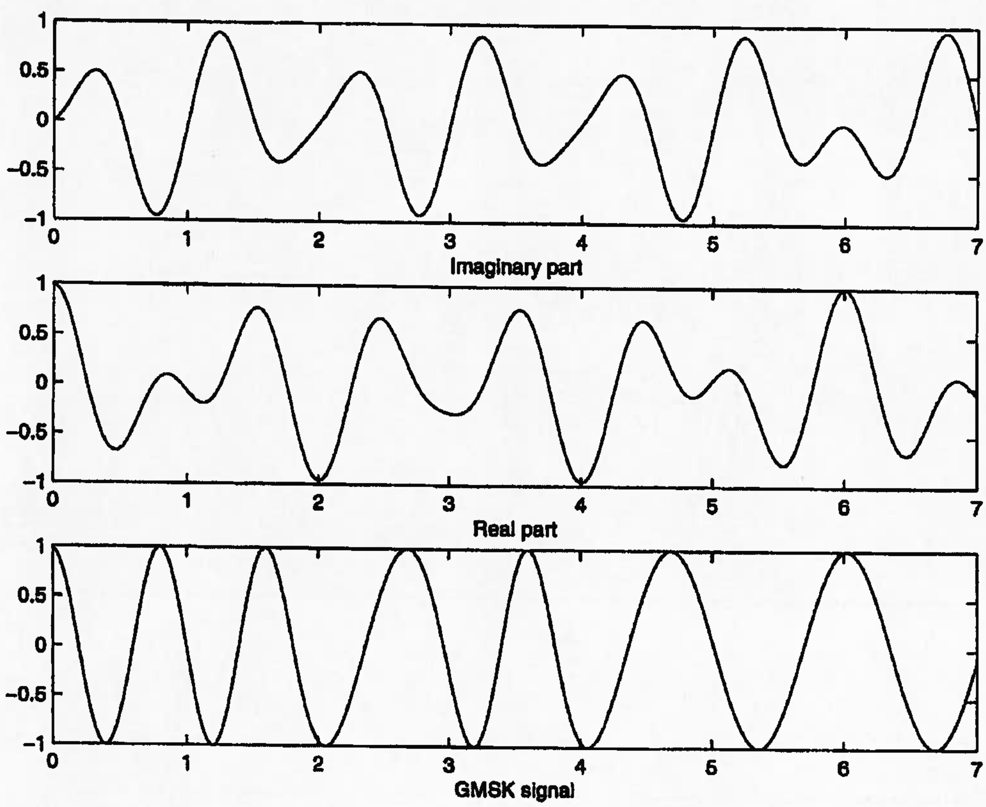
$$= -\frac{1}{2}[1 - \mathrm{erfc}(k_2)] + \frac{1}{2}[1 - \mathrm{erfc}(k_1)]$$

$$= \frac{1}{2}\mathrm{erfc}(k_2) - \frac{1}{2}\mathrm{erfc}(k_1)$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\frac{\pi(t - T/2)}{\alpha}\right] - \mathrm{erfc}\left[\frac{\pi(t - T/2)}{\alpha}\right]\right\}$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T} - \frac{1}{2}\right)\right] - \mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T} + \frac{1}{2}\right)\right]\right\}$$

Problem 2

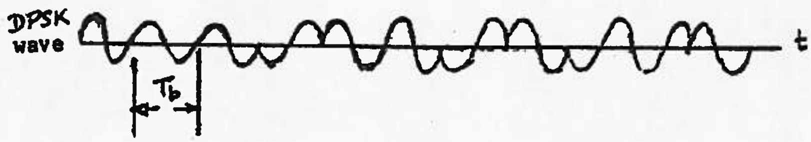


Problem : 3

(a)

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_k$ | | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| $d_{k-1}$ | | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| $d_k$ | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| Transmitted phase | 0 | 0 | 0 | $\pi$ | 0 | 0 | $\pi$ | 0 | $\pi$ | $\pi$ | 0 |

The waveform of the DPSK signal is thus as follows:




(b) Let

$x_I$ = output of the integrator in the in-phase channel

$x_Q$ = output of the integrator in the quadrature channel

$x_I'$ = one-bit delayed version of $x_I$

$x_Q'$ = one-bit delayed version of $x_Q$

$l_I$ = in-phase channel output

$= x_I x_I'$

$l_Q$ = quadrature channel output

$= x_Q x_Q'$

$y = l_I + l_Q$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Transmitted phase (radians) | 0 | 0 | 0 | $\pi$ | 0 | 0 | $\pi$ | 0 | $\pi$ | $\pi$ | 0 |
| Polarity of $x_I$ | + | + | + | - | + | + | - | + | - | - | + |
| Polarity of $x_I'$ | | + | + | + | - | + | + | - | + | - | - |
| Polarity of $l_I$ | | + | + | - | - | + | - | - | - | + | - |
| Polarity of $x_Q$ | - | - | - | + | - | - | + | - | + | + | - |
| Polarity of $x_Q'$ | | - | - | - | + | - | - | + | - | + | + |
| Polarity of $l_Q$ | | + | + | - | - | + | - | - | - | + | - |
| Polarity of y | | + | + | - | - | + | - | - | - | + | - |
| Reconstructed data stream | | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |

Problem 4

(a) For a coherent PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \tag{1}$$

For a DPSK system, we have

$$P_e = \frac{1}{2}\exp[-(E_b/N_0)_2] \tag{2}$$

Let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then, we may use Eqs. (1) and (2) to obtain

$$\sqrt{\pi}\,\sqrt{(E_b/N_0)_1} = \exp \delta$$

We are given that

$$\left(\frac{E_b}{N_0}\right)_1 = 7.2$$

Hence,

$$\delta = \ln[\sqrt{7.2\pi}]$$

$$= 1.56$$

Therefore,

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10} 7.2 = 8.57 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(7.2 + 1.56)$$

$$= 9.42 \text{ dB}$$

The separation between the two $(E_b/N_0)$ ratios is therefore 9.42 - 8.57 = 0.85 dB.

(b) For a coherent PSK system, we have

$$P_e = \frac{1}{2}\,\text{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \tag{3}$$

For a QPSK system, we have

$$P_e = \text{erfc}[\sqrt{(E_b/N_0)_2}]$$

$$\simeq \frac{\exp[-(E_b/N_0)_2]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_2}} \tag{4}$$

Here again, let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then we may use Eqs. (3) and (4) to obtain

$$\frac{1}{2} = \frac{\exp(-\delta)}{\sqrt{1 + \delta/(E_b/N_0)_1}} \tag{5}$$

Taking logarithms of both sides:

$$-\ln 2 = -\delta - 0.5 \ln[1 + \delta/(E_b/N_0)_1]$$

$$\simeq -\delta - 0.5 \frac{\delta}{(E_b/N_0)_1}$$

Solving for $\delta$:

$$\delta \simeq \frac{\ln 2}{1 + 0.5/(E_b/N_0)_1}$$

$$= 0.65$$

Therefore,

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(7.2) = 8.57 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(7.2 + .65)$$

$$= 8.95 \text{ dB.}$$

The separation between the two $(E_b/N_0)$ ratios is 8.95 - 8.57 = 0.38 dB.

(c) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$\simeq \frac{1}{2} \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \tag{6}$$

For a noncoherent binary FSK system, we have

$$P_e = \frac{1}{2} \exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right) \tag{7}$$

Hence,

$$\sqrt{\frac{\pi}{2}\left(\frac{E_b}{N_0}\right)_1} = \exp\left(\frac{\delta}{2}\right) \tag{8}$$

We are given that $(E_b/N_0)_1 = 13.5$. Therefore,

$$\delta = \ln(\frac{13.5\ \pi}{2})$$

$$= 3.055$$

We thus find that

$$10\ \log_{10}(\frac{E_b}{N_0})_1 = 10\ \log_{10}(13.5)$$

$$= 11.3\ \text{dB}$$

$$10\ \log_{10}(\frac{E_b}{N_0})_2 = 10\ \log_{10}(13.5 + 3.055)$$

$$= 12.2\ \text{dB}$$

Hence, the separation between the two $(E_b/N_0)$ ratios is 12.2 - 11.3 = 0.9 dB.

(d) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2}\ \text{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$\simeq \frac{1}{2}\ \frac{\exp(-\frac{1}{2}(\frac{E_b}{N_0})_1)}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_1}} \qquad (9)$$

For a MSK system, we have

$$P_e = \frac{1}{2}\ \text{erfc}[\sqrt{(E_b/2N_0)_2}] \qquad (10)$$

$$\simeq \frac{\exp[-\frac{1}{2}(\frac{E_b}{N_0})_2]}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_2}} \qquad (10)$$

Hence, using Eqs. (9) and (10), we

$$\ln 2 - \frac{1}{2}\ \ln[1 + \frac{\delta}{(E_b/N_0)_1}] = \frac{1}{2}\ \delta \qquad (11)$$

Noting that

$$\frac{\delta}{(E_b/N_0)_1} << 1$$

we may approximate Eq. (11) to obtain

$$\ln 2 - \frac{1}{2}\ [\frac{\delta}{(E_b/N_0)_1}] \simeq \frac{1}{2}\ \delta \qquad (11)$$

Solving for $\delta$, we obtain

$$\delta = \frac{2 \ln 2}{1 + \frac{1}{(E_b/N_0)_1}}$$

$$= \frac{2 \times 0.693}{1 + \frac{1}{13.5}}$$

$$= 1.29$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5) = 10 \times 1.13 = 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 1.29) = 11.7 \text{ dB}$$

Therefore, the separation between the two $(E_b/N_0)$ ratios is 11.7 - 11.3 = 0.4 dB.

Problem 5

The average power for any modulation scheme is

$$P = \frac{E_b}{T_b} .$$

This can be demonstrated for the three types given by integrating their power spectral densities from $-\infty$ to $\infty$,

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} [S_B(f - f_o) + S_B(f + f_o)] df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} S_B(f) df .$$

The baseband power spectral densities for each of the modulation techniques are:

| | PSK | QPSK | MSK |
|---|---|---|---|
| $S_B(f)$ | $2E_b \operatorname{sinc}^2(fT_b)$ | $4E_b \operatorname{sinc}^2(2fT_b)$ | $\frac{32E_b}{\pi^2}\left[\frac{\cos(2\pi fT_b)}{16f^2T_b^2 - 1}\right]^2$ |

Since $\int_{-\infty}^{\infty} a \operatorname{sinc}^2(ax)dx = 1$, $P = \frac{E_b}{T_b}$ is easily derived for PSK and QPSK. For MSK we have