Assignment 5

1. Let $Y = A\cos(\omega t) + c$ where A has mean m and variance σ^2 and ω and c are constants. Find the mean and variance of Y. compare the results to those obtained in following example.
   Example: Expected Values of a Sinusoid with Random Phase
   Let $Y = a\cos(\omega t + \Theta)$ where a, ω, and t are constants, and Θ is a uniform random variable in the interval$(0, 2\pi)$. The random variable Y results from sampling the amplitude of a sinusoid with random phase Θ. Find the expected value of Y and expected value of the power of Y, Y^2.

$$\begin{aligned} E[Y] &= E[a\cos(\omega t + \Theta)] \\ &= \int_0^{2\pi} \cos(\omega t + \theta)\frac{d\theta}{2\pi} = -a\sin(\omega t + \theta)\Big|_0^{2\pi} \\ &= -a\sin(\omega t + 2\pi) + a\sin(\omega t) = 0 \end{aligned}$$

   The average power is

$$\begin{aligned} E[Y^2] &= E[a^2\cos^2(\omega t + \Theta)] = E\left[\frac{a^2}{2} + \frac{a^2}{2}\cos(2\omega t + 2\Theta)\right] \\ &= \frac{a^2}{2} + \frac{a^2}{2}\int_0^{2\pi} \cos(2\omega t + \theta)\frac{d\theta}{2\pi} = \frac{a^2}{2} \end{aligned}$$

   Note that these answers are in agreement with the time averages of sinusoid: the time average ("dc" value) of the sinusoid is zero; the time-average power is $a^2/2$.

2. Find the mean and variance of the Gaussian random variable by applying the moment theorem to the characteristic function:

$$\Phi_X(\omega) = e^{jm\omega - \sigma^2\omega^2/2}$$

3. (a) Find the probability generating function of the geometric random variable.
   (b) Find the mean and variance of the geometric random variable from its pgf.

4. Let X be a discrete random variable with entropy H_X.
   (a) Find the entropy of $Y = 2X$.

(b) Find the entropy of any invertible transformation of X.

5. Let X take on value from $\{1, 2, \ldots, K\}$. Suppose that $P[X = K] = p$, and let H_Y be the entropy of X given that X is not equal to K. Show that $H_X = -p \ln p - (1-p)\ln(1-p) + (1-p)H_Y$.

6. A communication channel accepts as input either 000 or 111. The channel transmits each binary input correctly with probability $1 - p$ and erroneously with probability p. Find the entropy of the input given that the output is 000; given that the output is 010.