Assignment 6

1. Let X and Y have joint pdf:

$$f_{X,Y}(x,y) = k(x+y) \qquad\qquad \texttt{for } 0 \le x \le 1\texttt{, } 0 \le y \le 1.$$

   (a) Find k.

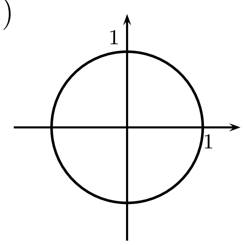
   (b) Find the joint cdf of (X,Y).

   (c) Find the marginal pdf of X and of Y.

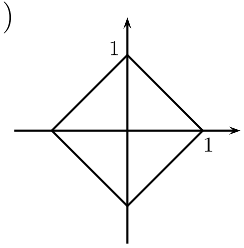
   (d) Find $P[X < Y]$, $P[Y < X^2]$, $P[X + Y > 0.5]$.

2. The random vector (X, Y) is uniformly distributed (i.e., $f(x, y) = k$) in the regions shown in the following figures and zero elsewhere.

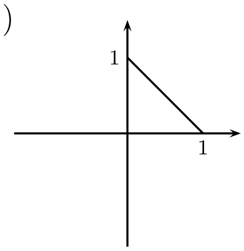
(i)



(ii)



(iii)



   (a) Find the value of k in each case.

   (b) Find the marginal pdf for X and for Y in each case.

   (c) Find $P[X > 0, Y > 0]$.

3. Let X and Y be independent random variable. Find the expression for the probability of the following events in terms of $F_X(x)$ and $F_Y(y)$.

   (a) $\{a < X \le b\} \cap \{Y > d\}$.

   (b) $\{a < X \le b\} \cap \{c \le Y < d\}$.

   (c) $\{|X| < a\} \cap \{c \le Y \le d\}$.

4. Let X and Y be the jointly Gaussian random variables with the means m_1 and m_2 and variances σ_1 and σ_2 respectively. The pdf is as following:

$$f_{X,Y}(x,y) = \frac{exp\left\{\frac{-1}{2(1-\rho_{X,Y}^2)}[(\frac{x-m_1}{\sigma_1})^2 - 2\rho_{X,Y}(\frac{x-m_1}{\sigma_1})(\frac{y-m_2}{\sigma_2}) + (\frac{y-m_2}{\sigma_2})^2]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{X,Y}^2}}$$

(a) Show the X and Y are independent random variables if and only if $\rho = 0$.

(b) Suppose $\rho = 0$, find $P[XY < 0]$.

5. (a) Find $E[(X+Y)^2]$.

(b) Find the variance of $X + Y$.

(c) Under what condition is the variance of the sum equal to the sum of the individual variances?

6. (a) Find $f_Y(y|x)$ in Problem 2(i).

(b) Find $E[Y|X = x]$ and $E[Y]$.

(c) Repeat parts (a) and (b) in Problem 2(ii).

(d) Repeat parts (a) and (b) in Problem 2(iii).

7. A message requires N time units to be transmitted, where N is a geometric random variable with pmf $p_i = (1-a)a^{i-1}, i = 1, 2, ...$ A single new message arrives during a time unit with probability p, and no messages arrive with probability $1 - p$. Let K be the number of new messages that arrive during the transmission of a single message.

(a) Find $E[K]$ and VAR[K] using conditional expectation.

(b) Find the pmf of K. *hint:*$(1-\beta)^{-(k+1)} = \sum_{n=k}^{\infty} \binom{n}{k} \beta^{n-k}$.

(c) Find the conditional pmf of N given by $K = k$.

(d) Find the value of n that maximizes $P[N = n|X = x]$.

8. The random variables X and Y have the joint pdf

$$f_{X,Y}(x,y) = e^{-(x+y)} \qquad \texttt{for } 0 < y < x < 1.$$

Find the pdf of $Z = X + Y$.

9. Let X and Y be jointly Gaussian random variable with pdf

$$f_{X,Y}(x,y) = \frac{exp\{\frac{-1}{2}[x^2 + 4y^2 - 3xy + 3y - 2x + 1]\}}{2\pi} \qquad \texttt{for all } x, y.$$

Find $E[X]$,$E[Y]$,VAR[X],VAR[Y], and COV(X, Y).

10. Let X and Y be zero-mean, independent Gaussian random variables with $\sigma^2 = 1$.

   (a) Find the value of r for which the probability that (X, Y) falls inside a circle of radius r is $1/2$.

   (b) Find the conditional pdf of (X, Y) given that (X, Y) is not inside a ring with inner radius r_1 and outer radius r_2.