Assignment 7

1. Let X, Y, Z have joint pdf

$$f_{X,Y,Z}(x, y, z) = k(x + y + z) \qquad \texttt{for } 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.$$

   (a) Find k.

   (b) Find $f_X(x|y, z)$ and $f_Z(z|x, y)$.

   (c) Find $f_X(x)$, $f_Y(y)$, and $f_Z(z)$.

2. Show that $f_{X,Y,Z}(x, y, z) = f_Z(z|x, y)f_Y(y|x)f_X(x)$.

3. Let U_1,U_2 and U_3 be independent random variables and let $X = U_1, Y = U_1 + U_2$, and $Z = U_1 + U_2 + U_3$.

   (a) Use the result in Problem 2 to find the joint pdf of X, Y, and Z.

   (b) Let the U_i be independent uniform random variables in the interval [0,1]. Find the marginal pdf of Y and Z. Find the marginal pdf of Z.

   (c) Let the U_i be independent zero-mean, unit variance Gaussian random variables. Find the marginal pdf of Y and Z. Find the marginal pdf of Z.

4. A random experiment has four possible outcomes. Suppose that the experiment is repeated n independent times and let X_k be the number of times outcome k occurs. The joint pmf of (X_1, X_2, X_3) is given by

$$p(k_1, k_2, k_3) = \frac{n!3!}{(n+3)!} = \binom{n+3}{3}^{-1} \qquad \texttt{for } 0 < k_i \texttt{ and } k_1 + K_2 + k_3 \leq n.$$

   (a) Find the marginal pmf of (X_1, X_2).

   (b) Find the marginal pmf of X_1.

   (c) Find the condition joint pmf of (X_2, X_3) given $X_1 = m$, where $0 \leq m \leq n$.

5. Let X, Y and Z be independent zero-mean, unit variance Gaussian random variables.

   (a) Find the pdf of $R = (X^2 + Y^2 + Z^2)^{1/2}$.

(b) Find the pdf of $R^2 = X^2 + Y^2 + Z^2$.

6. Let $W = aX + bY + c$, where X and Y are random variables.

   (a) Find the characteristic function of W in terms of the joint characteristic function of X and Y.

   (b) Find the characteristic function of W if X and Y are the random variables discussed in the following example. Find the pdf of W.

Example:
Suppose U and V are independent zero-mean, unit-variance Gaussian random variables, and let

$$X = U + V \qquad\qquad Y = 2U + V$$

Find the joint characteristic function of X and Y, and find $E[XY]$. The joint characteristic function of X and Y is

$$\begin{aligned}\Phi_{X,Y}(\omega_1, \omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] = E[e^{j\omega_1(U+V)} e^{j\omega_2(2U+V)}] \\ &= E[e^{j(\omega_1+2\omega_2)U + (\omega_1+\omega_2)V}].\end{aligned}$$

Since U and V are independent random variables, the joint characteristic function of U and V is equal to the product of the marginal characteristic functions:

$$\begin{aligned}\Phi_{X,Y}(\omega_1, \omega_2) &= E[e^{j((\omega_1+2\omega_2)U)}]E[e^{j((\omega_1+\omega_2)V)}] \\ &= \Phi_U(\omega_1 + 2\omega_2)\Phi_V(\omega_1 + \omega_2) \\ &= e^{-\frac{1}{2}(\omega_1+2\omega_2)^2} e^{-\frac{1}{2}(\omega_1+\omega_2)^2} \\ &= e^{\{-\frac{1}{2}(2\omega_1^2+6\omega_1\omega_2+5\omega_2^2)\}}.\end{aligned}$$

The correlation $E[XY]$ is found from equation

$$E[X^i Y^k] = \frac{1}{j^{i+k}} \frac{\partial^i \partial^k}{\partial \omega_1^i \partial \omega_2^k} \Phi_{X,Y}(\omega_1, \omega_2)|_{\omega_1=0,\omega_2=0}$$

with $i = 1$ and $k = 1$:

$$\begin{aligned}E[XY] &= \frac{1}{j^2} \frac{\partial^2}{\partial \omega_1 \partial \omega_2} \Phi_{X,Y}(\omega_1, \omega_2)|_{\omega_1=0,\omega_2=0} \\ &= -e^{\{-\frac{1}{2}(2\omega_{12}+6\omega_1\omega_2+5\omega_{22})\}}[6\omega_1 + 10\omega_2]\left(\frac{1}{4}\right)[4\omega_1 + 6\omega_2] \\ &+ \frac{1}{2} e^{\{-\frac{1}{2}(2\omega_1^2+6\omega_1\omega_2+5\omega_2^2)\}}[6]|_{\omega_1=0,\omega_2=0} = 3\end{aligned}$$

You should verify this answer by evaluating $E[XY] = E[(U + V)(2U + V)]$ directly.