Assignment 8

1. Let $X_1, ..., X_n$ be random variables with the same mean and with covariance function:

$$COV(X_i, X_j) = \begin{cases} \sigma^2 & \text{if } i = j, \\ \rho\sigma^2 & \text{if} |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

2. Let $X_1, ..., X_n$ be random variables with the same mean and with covariance function

$$COV(X_i, X_j) = \sigma^2 \rho^{|i-j|},$$

   where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

3. Let $Z = 3X - 7Y$, where X and Y are independent random variables.
   (a) Find the characteristic function of Z.
   (b) Find the mean and variance of Z by taking derivatives of the characteristic function found in part a.

4. Suppose that the number of particle emissions by a radioactive mass in t seconds is a Poisson random variable with mean λt. Use the Chebyshev inequality to obtain a bound for the probability that $|N(t)/t - \lambda|$ exceeds ϵ.

5. A fair die is tossed 20 times. Use the following equation to bound the probability that the total number of dots is between 60 and 80.

$$P[|M_n - \mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

6. Does the weak law of large numbers hold for the sample mean if the X_i's have the covariance functions given in Problem 1? Assume the X_i have the same mean.

7. (a) A fair coin is tossed 100 times. Estimate the probability that the number of heads is between 40 and 60. Estimate the probability that the number is between 50 and 55.

   (b) Repeat part a for $n = 1000$ and the intervals $[400, 600]$ and $[500, 550]$.

8. The number of messages arriving at a multiplexer is a Poisson random variable with mean 15 messages/second. Use the central limit theorem to estimate the probability that more than 950 message arrive in one minute.

9. A binary transmission channel introduces bit errors with probability 0.15. Estimate the probability that there are 20 or fewer errors in 100 bit transmissions.