Chapter 10
Analysis and Processing of Random Signals

ENCS6161 - Probability and Stochastic Processes

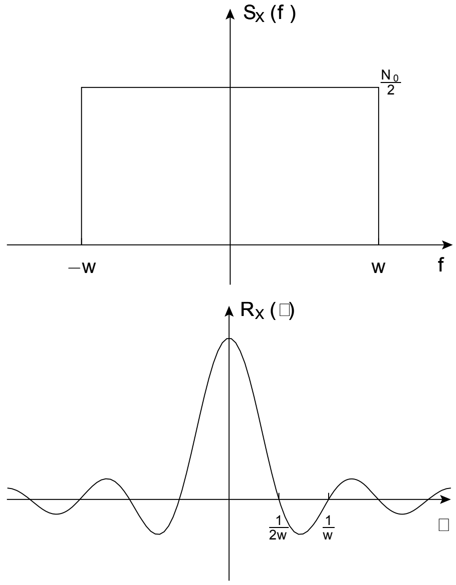
Concordia University

Power Spectral Density

- For WSS r.p. $X(t)$, the power spectral density (PSD)
$$S_X(f) \triangleq \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$
So, $R_X(\tau) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} df$
- The average power of $X(t)$
$$E[X(t)^2] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f) df$$
Note: $S_X(f) \geq 0$ (see pg. 412 of textbook for proof)
- Cross-Power Spectral Density:
Fro two WSS r.p. $X(t), Y(t)$
$$S_{X,Y}(f) = \mathcal{F}[R_{X,Y}(\tau)]$$
where $R_{X,Y}(\tau) = E[X(t+\tau)Y(t)]$.

Power Spectral Density

- Example: a WSS r.p. with $S_X(f) = \frac{N_0}{2}$ for $|f| < w$



$$E[X^2(t)] = \int_{-w}^{w} \frac{N_0}{2} df = N_0 w$$

$$R_X(\tau) = \int_{-w}^{w} \frac{N_0}{2} e^{j2\pi f\tau} df = \frac{N_0 \sin(2\pi w\tau)}{2\pi\tau}$$

If $\tau = \pm\frac{k}{2w}$, then $X(t)$ and $X(t+\tau)$ are uncorrelated.

Power Spectral Density

- White Noise: $S_X(f) = \frac{N_0}{2}$, for all f

$$\Rightarrow R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

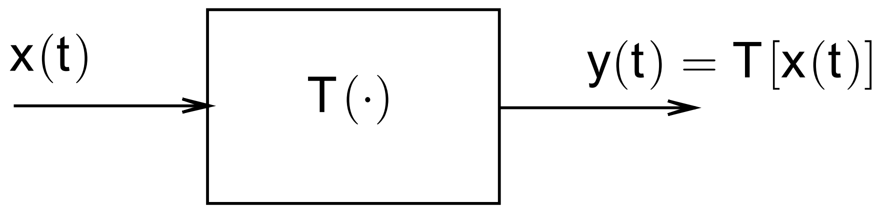
- Discrete-Time Random Processes

$$S_X(f) = \mathcal{F}[R_X(k)] = \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk}$$

$$R_X(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)e^{j2\pi fk}df$$

We only need to consider $-\frac{1}{2} < f < \frac{1}{2}$, since $S_X(f)$ is periodic in f with period of 1.

Linear Time-Invariant Systems



- A system is linear if,

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$$

- A system is time-invariant if,

$$y(t) = T[x(t)] \Rightarrow y(t-\tau) = T[x(t-\tau)]$$

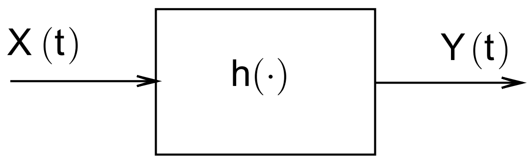
Response of LTI Systems

- $h(t) = T[\delta(t)]$ is called the Impulse Response of the system $T(\cdot)$.
- The response of $T(\cdot)$ to $x(t)$ will then be

$$\begin{aligned} y(t) &= T[x(t)] = \int_{-\infty}^{\infty} x(s)h(t-s)ds \\ &= \int_{-\infty}^{\infty} h(s)x(t-s)ds \\ &= x(t) * h(t) \end{aligned}$$

Response of LTI Systems

- Let's now consider the output of an LTI to a random WSS signal $X(t)$.



$$
\begin{aligned}
m_Y(t) &= E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds\right] \\
&= \int_{-\infty}^{\infty} h(s)E[X(t-s)]ds = m_X \int_{-\infty}^{\infty} h(s)ds
\end{aligned}
$$

Let $H(f) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt$, then

$$m_Y(t) = m_X H(0)$$

Response of LTI Systems

- The autocorrelation of the output $Y(t)$

$$
\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E\left[\int_{-\infty}^{\infty} h(s)X(t-s)ds \int_{-\infty}^{\infty} h(r)X(t+\tau-r)dr\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)E[X(t-s)X(t+\tau-r)]dsdr \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau+s-r)dsdr
\end{aligned}
$$

Response of LTI Systems

- The power spectral density

$$S_Y(f) = \int_{-\infty}^{\infty} R_Y(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(\tau + s - r)e^{-j2\pi f\tau} ds dr d\tau$$

(let $u = \tau + s - r$)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_X(u)e^{-j2\pi f(u-s+r)} ds dr du$$

$$= \int_{-\infty}^{\infty} h(s)e^{j2\pi fs} ds \int_{-\infty}^{\infty} h(r)e^{-j2\pi fr} dr \int_{-\infty}^{\infty} R_X(u)e^{-j2\pi fu} du$$

$$= H^*(f)H(f)S_X(f) = |H(f)|^2 S_X(f)$$

Response of LTI Systems

- Similarly,

$$R_{Y,X}(\tau) = E[Y(t+\tau)X(t)] = R_X(\tau) * h(\tau)$$

$$S_{Y,X}(f) = H(f)S_X(f)$$

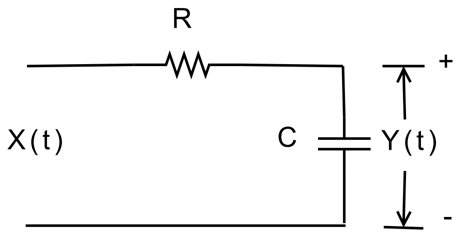
also,

$$S_{X,Y}(f) = S^*_{Y,X}(f) = H^*(f)S_X(f)$$

since $R_{X,Y}(\tau) = R_{Y,X}(-\tau)$

Response of LTI Systems

- Example: A white Gaussian Signal $X(t)$ is applied to an RC circuit. Find the average power and autocorrelation of the output $Y(t)$.



$$x(t) = RC\frac{dy}{dt} + y(t)$$

$$\Rightarrow \quad x(f) = j2\pi fRCy(f) + y(f)$$

$$\Rightarrow \quad H(f) = \frac{y(f)}{x(f)} = \frac{1}{1 + j2\pi fRC}$$

Response of LTI Systems

- $$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) = \left| \frac{1}{1 + j2\pi f RC} \right|^2 \frac{N_0}{2} \\ &= \frac{N_0/2}{1 + 4\pi^2 f^2 R^2 C^2} \\ R_Y(\tau) &= \mathcal{F}^{-1}[S_Y(f)] = \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}} \end{aligned}$$

  Average Power: $E[Y^2(t)] = R_Y(0) = \frac{N_0}{4RC}$