Chapter 11
Markov Chains

ENCS6161 - Probability and Stochastic Processes

Concordia University

Markov Processes

- A Random Process is a Markov Process if the future of the process given the present is independent of the past, i.e., if $t_1 < t_2 < \cdots < t_k < t_{k+1}$, then

$$
\begin{aligned}
& P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k, \cdots, X(t_1) = x_1] \\
= \quad & P[X(t_{k+1}) = x_{k+1} | X(t_k) = x_k]
\end{aligned}
$$

  if $X(t)$ is discreete-valued or

$$
\begin{aligned}
& f_{X(t_{k+1})}(x_{k+1} | X(t_k) = x_k, \cdots, X(t_1) = x_1) \\
= \quad & f_{X(t_{k+1})}(x_{k+1} | X(t_k) = x_k)
\end{aligned}
$$

  if $X(t)$ is continuous-valued.

Markov Processes

- Example: $S_n = X_1 + X_2 + \cdots + X_n$

$$\Rightarrow S_{n+1} = S_n + X_{n+1}$$

$$\begin{aligned} & P[S_{n+1} = s_{n+1} | S_n = s_n, \cdots, S_1 = s_1] \\ = & \quad P[S_{n+1} = s_{n+1} | S_n = s_n] \end{aligned}$$

  So S_n is a Markov process.

- Example: The Poisson process is a continuous-time Markov process.

$$\begin{aligned} & P[N(t_{k+1}) = j | N(t_k) = i, \cdots, N(t_1) = x_1] \\ = & \quad P[j - i \text{ events in } t_{k+1} - t_k] \\ = & \quad P[N(t_{k+1}) = j | N(t_k) = i] \end{aligned}$$

- An integer-valued Markov process is called Markov Chain.

Discrete-time Markov Chain

- X_n is a discrete-time Markov chain starts at $n = 0$ with
$$P_i(0) = P[X_0 = i], \quad i = 0, 1, 2, \cdots$$
Then from the Markov property,
$$P[X_n = i_n, \cdots, X_0 = i_0]$$
$$= P[X_n = i_n | X_{n-1} = i_{n-1}] \cdots P[X_1 = i_1 | X_0 = i_0] P[X_0 = i_0]$$
where $P[X_{k+1} = i_{k+1} | X_k = i_k]$ is called the one-step state transition probability.
- If $P[X_{k+1} = j | X_k = i] = p_{ij}$ for all k, X_n is said to have homogeneous transition probabilities.
$$P[X_n = i_n, \cdots, X_0 = i_0] = P_{i_0}(0) p_{i_0, i_1} \cdots p_{i_{n-1}, i_n}$$

Discrete-time Markov Chain

- The process is completely specified by the initial pmf $P_{i_0}(0)$ and the transition matrix

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where for each row:

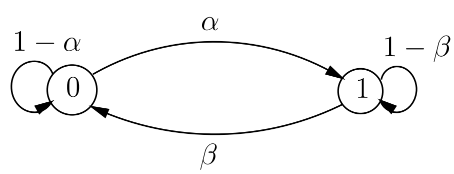
$$\sum_j p_{ij} = 1$$

Discrete-time Markov Chain

- Example: Two-state Markov Chain for speach activity (on-off source)

  two states: $\begin{cases} 0 & \text{silence (off)} \\ 1 & \text{with speach activity (on)} \end{cases}$

  State Transition Diagram



$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Discrete-time Markov Chain

- The n-Step Transition Probabilities

$$p_{ij}(n)\ ,\quad P[X_{k+n}=j|X_k=i]\quad n\geq 0$$

Let $P(n)$ be the n-step transition probability matrix, i.e.

$$P(n)=\begin{bmatrix} p_{00}(n) & p_{01}(n) & p_{02}(n) & \cdots \\ p_{10}(n) & p_{11}(n) & p_{12}(n) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Then $P(n)=P^n$, where P is the one-step transition probability matrix.

The State Probabilities

- Let $\underline{p}(n) = \{P_j(n)\}$ be the state prob. at time n then

$$P_j(n) = \sum_i P[X_n = j | X_{n-1} = i] P[X_{n-1} = i]$$
$$= \sum_i p_{ij} P_i(n-1)$$

i.e. $\underline{p}(n) = \underline{p}(n-1)P$.

- By recursion:

$$\underline{p}(n) = \underline{p}(n-1)P = \underline{p}(n-2)P^2 = \cdots = \underline{p}(0)P^n$$

Steady State Probabilities

- In many cases, when $n \to \infty$, the Markov chain goes to steady state, in which the state probabilities do not change with n anymore, i.e.,
  $$\underline{p}(n) \to \underline{\pi}, \text{ as } n \to \infty$$
  $\underline{\pi}$ is called the Stationary State pmf.
- If the steady state exists, then when n is large, we have
  $$\underline{p}(n) = \underline{p}(n-1) = \underline{\pi}$$
  $$\Rightarrow \quad \underline{\pi} = \underline{\pi} P$$
  (note: $\sum_i \pi_i = 1$)

Steady State Probabilities

- Example: Find the steady state pmf of the on-off source.

$$\underline{\pi} P = \underline{\pi},$$

$$\Rightarrow \quad [\pi_0, \pi_1] \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} = [\pi_0, \pi_1]$$

together with $\pi_0 + \pi_1 = 1$

$$\Rightarrow \pi_0 = \frac{\beta}{\alpha+\beta} \quad \pi_1 = \frac{\alpha}{\alpha+\beta}$$

Continuous-Time Markov Chains

- If $P[X(s+t)=j|X(s)=i]=p_{ij}(t),\ t\geq 0$ for all s, then the continuous-time Markov chain $X(t)$ has homogeneous transition prob.
- The transition rate of $X(t)$ entering state j from i is defined as

$$r_{ij} \triangleq p'_{ij}(t)|_{t=0} = \begin{cases} \lim_{\delta\to 0} \frac{p_{ij}(\delta)}{\delta} & i \neq j \\ \lim_{\delta\to 0} \frac{p_{ij}(\delta)-1}{\delta} & i = j \end{cases}$$

Note:

$$p_{ij}(0) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Continuous-Time Markov Chains

- From

$$\begin{cases} P_j(t+\delta) = \sum_i P_i(t)p_{ij}(\delta) \\ P_j(t) = \sum_i P_i(t)p_{ij}(0) \end{cases}$$

We can show that:

$$\frac{P_j(t+\delta) - P_j(t)}{\delta} = \sum_i P_i(t)\frac{p_{ij}(\delta) - p_{ij}(0)}{\delta}$$

Let $\delta \to 0$, we have:

$$P_j'(t) = \sum_i P_i(t)r_{ij}$$

This is called Chapman-Kolmogorov equations.

Steady State Probabilities

- In the steady-state, $P_j(t)$ doesn't change with t, so

$$P_j'(t) = 0$$

and hence from Chapman-Kolmogorov equations

$$\sum_i P_i r_{ij} = 0 \quad \text{for all } j$$

These are called the Global Balancee Equations.