Chapter 7
Sums of Random Variables and Long-Term Averages

ENCS6161 - Probability and Stochastic Processes

Concordia University

Sums of Random Variables

- Let $X_1, \cdots, X_n$ be r.v.s and $S_n = X_1 + \cdots + X_n$, then

$$\begin{aligned} E[S_n] &= E[X_1] + \cdots + E[X_n] \\ Var[S_n] &= Var[X_1 + \cdots + X_n] \\ &= E\left[\sum_{i=1}^{n}(X_i - \mu_{X_i})\sum_{j=1}^{n}(X_j - \mu_{X_j})\right] \\ &= \sum_{i=1}^{n} Var[X_i] + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} Cov(X_i, X_j) \end{aligned}$$

- If $Z = X + Y \ (n = 2)$,

$$Var[Z] = Var[X] + Var[Y] + 2Cov(X, Y)$$

Sums of Random Variables

- Example: Sum of n i.i.d r.v.s with mean μ and variance σ^2.

$$\begin{aligned} E[S_n] &= E[X_1] + \cdots + E[X_n] = n\mu \\ Var[S_n] &= nVar[X_i] = n\sigma^2 \end{aligned}$$

- pdf of sums of independent random variables
  $X_1, \cdots, X_n$ indep r.v.s and $S_n = X_1 + \cdots + X_n$, then

$$\begin{aligned} \Phi_{S_n}(w) &= E[e^{jwS_n}] = E[e^{jw(X_1+\cdots+X_n)}] \\ &= \Phi_{X_1}(w) \cdots \Phi_{X_n}(w) \end{aligned}$$

  and

$$f_{S_n}(s) = \mathcal{F}^{-1}\left\{\Phi_{X_1}(w) \cdots \Phi_{X_n}(w)\right\}$$

Sums of Random Variables

- Example: $X_1 \cdots X_n$ indep and $X_i \sim N(m_i, \sigma_i^2)$. What is the pdf of $S_n = X_1 + \cdots + X_n$?
  For a Guassian r.v.
  $$X \sim N(\mu, \sigma^2) \Rightarrow \Phi_X(w) = e^{jw\mu - \frac{w^2\sigma^2}{2}}$$
  (prove it by yourself)
  So
  $$\Phi_{S_n}(w) = \prod_{i=1}^{n} e^{jwm_i - \frac{w^2\sigma_i^2}{2}} = e^{jw(m_1+\cdots+m_n) - w^2(\sigma_1^2+\cdots+\sigma_n^2)/2}$$
  $$)\ \ S_n \sim N(m_1 + \cdots + m_n, \sigma_1^2 + \cdots + \sigma_n^2)$$
  What if $X_1, \cdots, X_n$ are not indep?? (hint: use $\underline{Y} = A\underline{X}$)

Sums of Random Variables

- pdf of i.i.d r.v.s

$$\Phi_{S_n}(w) = (\Phi_X(w))^n$$

- Example:Find the pdf of the sum of n i.i.d exponential r.v.s with parameter λ.

  $\Phi_X(w) = \frac{\lambda}{\lambda - jw}$ (see table 3.2 on page 101)

$$\Rightarrow \quad \Phi_{S_n}(w) = (\frac{\lambda}{\lambda - jw})^n$$

$$\Rightarrow \quad f_{S_n}(s) = \frac{\lambda e^{-\lambda s}(\lambda s)^{n-1}}{(n-1)!}, \ s > 0$$

  This is the so called m-Erlang r.v.

Sums of Random Variables

- When dealing with non-negative integer-valued r.v.s, we use the probability generating function:

$$G_N(z) = E[z^N] = \sum_n z^n P_N(n)$$

$$P_N(n) = \frac{1}{n!} \frac{d^n}{dz^n} G_N(z) \Big|_{z=0}$$

- For $N = X_1 + \cdots + X_n$ where X_i are independent.

$$\begin{aligned} G_N(z) &= E[z^{X_1+\cdots+X_n}] \\ &= E[z^{X_1}] \cdots E[z^{X_n}] \\ &= G_{X_1}(z) \cdots G_{X_n}(z) \end{aligned}$$

Sums of Random Variables

- Example: Find the pdf of the sum of n independent Bernoulli r.v.s with $p_0 = 1 - p = q$ and $p_1 = p$.

$$
\begin{aligned}
G_X(z) &= E[z^X] = q + pz \\
&\Rightarrow G_N(z) = (q + pz)^n \\
&\Rightarrow P_N(k) = \binom{n}{k} p^k q^{n-k},\ k = 0, 1 \cdots n
\end{aligned}
$$

(See Table 3.1)

$$\Rightarrow N \sim \text{Binomial}(n, p)$$

The Sample Mean

- Let X be a r.v. with mean μ and variance σ^2. $X_1; \cdots; X_n$ denote n independent, repeated measurement of X. That is, X_i's are i.i.d r.v.s with the same pdf as X. The sample mean is defined as

$$M_n = \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The mean and variance of the sample mean are

$$E[M_n] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \mu$$

$$Var[M_n] = E[(M_n - \mu)^2] = E\left[\left(\frac{S_n - E(S_n)}{n}\right)^2\right]$$

$$= \frac{1}{n^2}E[(S_n - E(S_n)^2)] = \frac{1}{n^2}Var[S_n] = \frac{\sigma^2}{n}$$

The Laws of Large Numbers

- From chebyshev inequality for any $\varepsilon > 0$
$$P\{|M_n - \mu| \geq \varepsilon\} \leq \frac{Var[M_n]}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$
So $P\{|M_n - \mu| < \varepsilon\} > 1 - \frac{\sigma^2}{n\varepsilon^2}$
- The weak law of large numbers:
$$\lim_{n\to\infty} P\{|M_n - \mu| < \varepsilon\} = 1$$
for any $\varepsilon > 0$.
- The strong law of large numbers:
$$P\left\{\lim_{n\to\infty} M_n = \mu\right\} = 1$$
The proof is beyond the level of this course.

The Central Limit Theorem

- Let $X_1, \cdots, X_n$ be i.i.d r.v.s with μ, σ^2 and $S_n = X_1 + \cdots + X_n$. Let
$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$
then as $n \to \infty$, the distribution of Z_n tends to standard Gaussian.
$$\begin{aligned} \lim_{n\to\infty} P[Z_n \leq z] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \\ &= 1 - Q(z) = \Phi(z) \end{aligned}$$

The Central Limit Theorem

- Proof:

$$\Phi_{Z_n}(w) = E[e^{jwZ_n}] = E\left[e^{\frac{jw}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu)}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{\frac{jw(X_i-\mu)}{\sigma\sqrt{n}}}\right] = \prod_{i=1}^{n} E\left[e^{\frac{jw(X_i-\mu)}{\sigma\sqrt{n}}}\right] \quad \text{(* indep)}$$

$$= \left\{E\left[e^{\frac{jw(X_i-\mu)}{\sigma\sqrt{n}}}\right]\right\}^n \quad \text{(* i.i.d)}$$

(to be continued)

The Central Limit Theorem

- Proof: (continues)

$$E\left[e^{\frac{jw(X_i-\mu)}{\sigma\sqrt{n}}}\right] = E\left[1+\frac{jw}{\sigma\sqrt{n}}(X-\mu)+\frac{(jw)^2}{2!n\sigma^2}(X-\mu)^2+R(w)\right]$$

$$= 1+\frac{jw}{\sigma\sqrt{n}}\underbrace{E[X-\mu]}_{=0}-\frac{w^2}{2n\sigma^2}\underbrace{E[(X-\mu)^2]}_{=\sigma^2}+E[R(w)]$$

$$= 1-\frac{w^2}{2n}+E[R(w)]$$

$E[R(w)]$ becomes negligible compared to $\frac{w^2}{2n}$ when $n\to\infty$, therefore

$$\lim_{n\to\infty}\Phi_{Z_n}(w)=\lim_{n\to\infty}\left(1-\frac{w^2}{2n}\right)^n=e^{-\frac{w^2}{2}}$$

So, when $n\to\infty$, $Z_n \sim N(0,1)$

Convergence of Sequence of R.V.s

- $X_1, \cdots X_n$ are r.v.s, how to define the convergence of of r.v.s? Recall: a r.v. is a function: $S \to R$. So $X_1(w), X_2(w), \cdots$ are functions.
- If $X_n(w) \to X(w)$ for all w, sure convergence
- If $P\{w | X_n(w) \to X(w)\} = 1$, almost sure convergence, $X_n \to X$ a.s. (or w.p. 1)
- If $E[(X_n(w) - X(w))^2] \to 0$ as $n \to \infty$ mean square convergence, $X_n \to X$ m.s.
- If $\forall \varepsilon > 0$, $P\{|X_n(w) - X(w)| > \varepsilon\} \to 0$, convergence in probability.

Convergence of Sequence of R.V.s

- a.s. convergence $\Rightarrow$ convergence in probability
  m.s. convergence $\Rightarrow$ convergence in probability
  But almost sure $<\neq>$ mean square.
- Convergence in distribution
  X_n has cdf $F_n(x)$ and X has cdf $F(x)$. If $F_n(x) \rightarrow F(x)$ for all x where $F(x)$ is continuous. We call X_n converge to X in distribution.
- Convergence in prob. $\Rightarrow$ convergence in distribution.

Convergence of Sequence of R.V.s

- Note:
  weak LLN: convergence in prob.
  $$M_n \to \mu \text{ in prob.}$$
  strong LLN: almost sure
  $$M_n \to \mu \; a.s.$$
  CLT: convergence in distribution
  $$Z_n \to Z \sim N(0,1) \text{ in dist.}$$
- In fact $M_n \to \mu \; m.s.$ since
  $E[(M_n - \mu)^2] = Var[M_n] = \frac{\sigma^2}{n} \to 0$, as $n \to \infty$