Chapter 9
Random Processes

ENCS6161 - Probability and Stochastic Processes

Concordia University

Definition of a Random Process

- Assume the we have a random experiment with outcomes w belonging to the sample set S. To each $w \in S$, we assign a time function $X(t, w)$, $t \in I$, where I is a time index set: discrete or continuous. $X(t, w)$ is called a random process.
- If w is fixed, $X(t, w)$ is a deterministic time function, and is called a realization, a sample path, or a sample function of the random process.
- If $t = t_0$ is fixed, $X(t_0, w)$ as a function of w, is a random variable.
- A random process is also called a stochastic process.

Definition of a Random Process

- Example: A random experment has two outcomes $w \in \{0, 1\}$. If we assign:
$$\begin{aligned} X(t,0) &= A\cos t \\ X(t,1) &= A\sin t \end{aligned}$$
where A is a constant. Then $X(t,w)$ is a random process.
- Usually we drop w and write the random process as $X(t)$.

Specifying a Random Process

- Joint distribution of time samples
  Let $X_1, \cdots, X_n$ be the samples of $X(t, w)$ obtained at $t_1, \cdots, t_n$, i.e. $X_i = X(t_i, w)$, then we can use the joint CDF
  $$F_{X_1 \cdots X_n}(x_1, \cdots, x_n) = P[X_1 \le x_1, \cdots, X_n \le x_n]$$
  or the joint pdf $f_{X_1 \cdots X_n}(x_1, \cdots, x_n)$ to describe a random process partially.
- Mean function:
  $$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$
- Autocorrelation function
  $$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1)X(t_2)}(x, y)dxdy$$

Specifying a Random Process

- Autocovariance function

$$
\begin{aligned}
C_X(t_1, t_2) &= E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))] \\
&= R_X(t_1, t_2) - m_X(t_1)m_X(t_2)
\end{aligned}
$$

a special case:

$$C_X(t, t) = E[(X(t) - m_X(t))^2] = Var[X(t)]$$

- The correlation coefficient

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)C_X(t_2, t_2)}}$$

- Mean and autocorrelation functions provide a partial description of a random process. Only in certain cases (Gaussian), they can provide a fully description.

Specifying a Random Process

- Example: $X(t) = A\cos(2\pi t)$, where A is a random variable.

$$\begin{aligned}
m_X(t) &= E[A\cos(2\pi t)] = E[A]\cos(2\pi t) \\
R_X(t_1, t_2) &= E[A\cos(2\pi t_1) \cdot A\cos(2\pi t_2)] \\
&= E[A^2]\cos(2\pi t_1)\cos(2\pi t_2) \\
C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \\
&= (E[A^2] - E[A]^2)\cos(2\pi t_1)\cos(2\pi t_2) \\
&= Var(A)\cos(2\pi t_1)\cos(2\pi t_2)
\end{aligned}$$

Specifying a Random Process

- Example: $X(t) = A\cos(wt + \Theta)$, where Θ is uniform in $[0, 2\pi]$, A and w are constants.

$$
\begin{aligned}
m_X(t) &= E[A\cos(wt+\Theta)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} A\cos(wt+\theta)d\theta = 0 \\
C_X(t_1,t_2) &= R_X(t_1,t_2) = A^2E[\cos(wt_1+\Theta)\cos(wt_2+\Theta)] \\
&= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{\cos w(t_1-t_2) + \cos[w(t_1+t_2)+\theta]}{2}d\theta \\
&= \frac{A^2}{2}\cos w(t_1-t_2)
\end{aligned}
$$

Gaussian Random Processes

- A random process $X(t)$ is a Gaussian random process if for any n, the samples taken at $t_1, t_2, \cdots, t_n$ are jointly Gaussian, i.e. if
$$X_1 = X(t_1), \cdots, X_n = X(t_n)$$
then
$$f_{X_1 \cdots X_n}(x_1, \cdots, x_n) = \frac{e^{-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})}}{(2\pi)^{n/2}|K|^{1/2}}$$
where $\underline{m} = [m_X(t_1), \cdots, m_X(t_n)]^T$ and
$$K = \begin{bmatrix} C_X(t_1,t_1) & \cdots & C_X(t_1,t_n) \\ \cdots & \cdots & \cdots \\ C_X(t_n,t_1) & \cdots & C_X(t_n,t_n) \end{bmatrix}$$

Multiple Random Processes

- To specify joint random processes $X(t)$ and $Y(t)$, we need to have the pdf of all samples of $X(t)$ and $Y(t)$ such as $X(t_1), \cdots, X(t_i), Y(t_1^{'}), \cdots, Y(t_j^{'})$ for all i and j and all choices of $t_1, \cdots, t_i, t_1^{'}, \cdots, t_j^{'}$.
- The processes $X(t)$ and $Y(t)$ are indepedent if the random vectors $(X(t_1), \cdots, X(t_i))$ and $(Y(t_1^{'}), \cdots, Y(t_j^{'}))$ are independent for all i, j and $t_1, \cdots, t_i, t_1^{'}, \cdots, t_j^{'}$.

Multiple Random Processes

- The cross-correlation $R_{X,Y}(t_1, t_2)$ is defined as
$$R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
Two processes are orthogonal if
$$R_{X,Y}(t_1, t_2) = 0 \text{ for all } t_1 \text{ and } t_2$$
- The cross-covariance
$$\begin{aligned} C_{X,Y}(t_1, t_2) &= E[(X(t_1) - m_X(t_1))(Y(t_2) - m_Y(t_2))] \\ &= R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2) \end{aligned}$$
$X(t)$ and $Y(t)$ are uncorrelated if
$$C_{X,Y}(t_1, t_2) = 0 \text{ for all } t_1 \text{ and } t_2$$

Multiple Random Processes

- Example: $X(t) = \cos(wt + \Theta)$ and $Y(t) = \sin(wt + \Theta)$, where Θ is uniform in $[0, 2\pi]$ and w is a constant.

$$
\begin{aligned}
m_X(t) &= m_Y(t) = 0 \\
C_{X;Y}(t_1, t_2) &= R_{X;Y}(t_1, t_2) \\
&= E[\cos(wt_1 + \Theta)\sin(wt_2 + \Theta)] \\
&= E\left[-\frac{1}{2}\sin w(t_1 - t_2) + \frac{1}{2}\sin(w(t_1 + t_2) + 2\Theta)\right] \\
&= -\frac{1}{2}\sin w(t_1 - t_2)
\end{aligned}
$$

Discrete-Time Random Processes

- i.i.d random processes: $X_n \sim f_X(x_n)$ then

$$F_{X_1 \cdots X_n}(x1, \cdots, x_n) = F_X(x1) \cdots F_X(x_n)$$

$$\begin{aligned}
m_X(n) &= E[X_n] = m \quad \text{for all } n \\
C_X(n_1, n_2) &= E[(X_{n_1} - m)(X_{n_2} - m)] \\
&= E[X_{n_1} - m]E[X_{n_2} - m] = 0 \text{ if } n_1 \neq n_2 \\
C_X(n, n) &= E[(X_n - m)^2] = \sigma^2 \\
\Rightarrow & \quad C_X(n_1, n_2) = \sigma^2 \delta_{n_1;n_2} \\
R_X(n_1, n_2) &= \sigma^2 \delta_{n_1;n_2} + m^2
\end{aligned}$$

Discrete-Time Random Processes

- Example: let X_n be a sequence of i.i.d. Bernoulli r.v.s. with $P(X_i = 1) = p$.

$$\begin{aligned} m_X(n) &= p \\ Var(X_n) &= p(1-p) \\ C_X(n_1, n_2) &= p(1-p)\delta_{n_1;n_2} \\ R_X(n_1, n_2) &= p(1-p)\delta_{n_1;n_2} + p^2 \end{aligned}$$

- Example:
  $Y_n = 2X_n - 1$, where X_n are i.i.d. Bernoulli r.v.s

$$Y_n = \begin{cases} 1 & \text{with} \quad p \\ -1 & \text{with} \quad (1-p) \end{cases}$$

$$\Rightarrow m_Y(n) = 2p - 1, \quad Var(Y_n) = 4p(1-p)$$

Random Walk

- Let $S_n = Y_1 + \cdots + Y_n$, where Y_n are i.i.d. r.v.s. with $P\{Y_n = 1\} = p$ and $P\{Y_n = -1\} = 1 - p$. This is a one-dimensional random walk.
- If there are k positive jumps ($+1$'s) in n trials (n walks), then there are $n - k$ negative jumps (-1's). So $S_n = k \times 1 + (n - k) \times (-1) = 2k - n$ and

$$P\{S_n = 2k - n\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n$$

Properties of Random Walk

- Independent increment

  Let $I_1 = (n_0, n_1]$ and $I_2 = (n_2, n_3]$. If $n_1 \leq n_2$, I_2 and I_2 do not overlap. Then the increments on the two intervals are

$$
\begin{aligned}
S_{n_1} - S_{n_0} &= Y_{n_0+1} + \cdots + Y_{n_1} \\
S_{n_3} - S_{n_2} &= Y_{n_2+1} + \cdots + Y_{n_3}
\end{aligned}
$$

  Since they have no Y_n's in common (no overlapping) and Y_n's are independent.
  $\Rightarrow S_{n_1} - S_{n_0}$ and $S_{n_3} - S_{n_2}$ are independent.
  This property is called independent increment.

Properties of Random Walk

- Stationary increment

  Furthermore, if I_1 and I_2 have the same length, i.e $n_1 - n_0 = n_3 - n_2 = m$, then the increments $S_{n_1} - S_{n_0}$ and $S_{n_3} - S_{n_2}$ have the same distribution since they both are the sum of m i.i.d r.v.s

  This means that the increments over interval of the same length have the same distribution. The process S_n is said to have stationary increment.

Properties of Random Walk

- These two properties can be used to find the joint pmf of S_n at $n_1, \cdots, n_k$

$$P[S_{n_1} = s_1, S_{n_2} = s_2, \cdots, S_{n_k} = s_k]$$

$$= P[S_{n_1} = s_1, S_{n_2} - S_{n_1} = s_2 - s_1, \cdots, S_{n_k} - S_{n_{k-1}} = s_k - s_{k-1}]$$

$$= P[S_{n_1} = s_1]P[S_{n_2} - S_{n_1} = s_2 - s_1] \cdots P[S_{n_k} - S_{n_{k-1}} = s_k - s_{k-1}]$$

(from independent increment)

$$= P[S_{n_1} = s_1]P[S_{n_2 - n_1} = s_2 - s_1] \cdots P[S_{n_k - n_{k-1}} = s_k - s_{k-1}]$$

(from stationary increment)

Properties of Random Walk

- If Y_n are continuous valued r.v.s.

$$f_{S_{n_1} \cdots S_{n_k}}(s_1, \cdots s_k)$$
$$= f_{S_{n_1}}(s_1) f_{S_{n_2 - n_1}}(s_2 - s_1) \cdots f_{S_{n_k - n_{k-1}}}(s_k - s_{k-1})$$

e.g., if $Y_n \sim N(0, \sigma^2)$ then

$$f_{S_{n_1}; S_{n_2}}(s_1, s_2) = f_{S_{n_1}}(s_1) f_{S_{n_2 - n_1}}(s_2 - s_1)$$
$$= \frac{1}{\sqrt{2\pi n_1}\sigma} e^{-\frac{s_1^2}{2n_1\sigma^2}} \cdot \frac{1}{\sqrt{2\pi(n_2 - n_1)}\sigma} e^{-\frac{(s_2 - s_1)^2}{2(n_2 - n_1)\sigma^2}}$$

Sum of i.i.d Processes

- If $X_1, X_2, \ldots, X_n$ are i.i.d and $S_n = X_1 + X_2 + \ldots + X_n$, we call S_n the sum process of i.i.d, e.g. random walk is a sum process.

$$
\begin{aligned}
m_S(n) &= E[S_n] = nE[X] = nm \\
Var[S_n] &= nVar[X] = n\sigma^2
\end{aligned}
$$

Autocovariance

$$
\begin{aligned}
C_S(n,k) &= E[(S_n - E[S_n])(S_k - E[S_k])] \\
&= E[(S_n - nm)(S_k - km)] = E\left[\sum_{i=1}^{n}(X_i - m)\sum_{j=0}^{k}(X_j - m)\right] \\
&= \sum_{i=1}^{n}\sum_{j=0}^{k} E[(X_i - m)(X_j - m)] = \sum_{i=1}^{n}\sum_{j=0}^{k} C_X(i,j) \\
&= \sum_{i=1}^{n}\sum_{j=0}^{k} \sigma^2 \delta_{ij} = \min(n,k)\sigma^2
\end{aligned}
$$

Sum of i.i.d Processes

- Example: For random Walk

$$
\begin{aligned}
E[S_n] &= nm = n(2p-1) \\
Var[S_n] &= n\sigma^2 = 4np(1-p) \\
C_S(n,k) &= \min(n,k)4p(1-p)
\end{aligned}
$$

Continuos Time Random Processes

- Poisson Process: a good model for arrival process
  $N(t)$: Number of arrivals in $[0, t]$
  λ : arrival rate (average # of arrivals per time unit)
  We divide $[0, t]$ into n subintervals, each with duration $\delta = \frac{t}{n}$
- Assume:
  - The probability of more than one arrival in a subinterval is negligible.
  - Whether or not an event (arrival) occurs in a subinterval is *independent* of arrivals in other subintervals.

  So the arrivals in each subinterval are *Bernoulli* and they are *independent*.

Poisson Process

- Let $p = Prob\{1$ arrival$\}$. Then the average number of arrivals in $[0, t]$ is

  $$np = \lambda t \quad \Rightarrow \quad p = \frac{\lambda t}{n}$$

  The total arrivals in $[0, t] \sim$ Bionomial(n, p)

  $$P[N(t) = k] = \binom{n}{k} p^k (1-p)^k \rightarrow \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

  when $n \rightarrow \infty$.

- Stationary increment? Yes
  Independent increment? Yes

Poisson Process

- Inter-arrival Time: Let T be the inter-arrival time.

$$\begin{aligned} P\{T > t\} &= P\{\text{no arrivals in } t \text{ seconds}\} \\ &= P\{N(t) = 0\} = e^{-\lambda t} \\ \Rightarrow \quad & P\{T \leq t\} = 1 - e^{-\lambda t} \\ & f_T(t) = \lambda e^{-\lambda t}, \text{ for } t \geq 0 \end{aligned}$$

So the inter-arrival time is exponential with mean $\frac{1}{\lambda}$.

Random Telegraph Signal

- Read on your own

Wiener Process

- Suppose that the symmetric random walk ($p = \frac{1}{2}$) takes steps of magnitude of h every δ seconds. At time t , we have $n = \frac{t}{\delta}$ jumps.

$$X_{\delta}(t) = h(D_1 + D_2 + \cdots + D_n) = hS_n$$

where D_i are *i.i.d* random variables taking ± 1 with equal probability.

$$E[X_{\delta}(t)] = hE[S_n] = 0$$

$$Var[X_{\delta}(t)] = h^2 n Var[D_i] = h^2 n$$

Wiener Process

- If we let $h = \sqrt{\alpha\delta}$, where α is a constant and $\delta \to 0$ and let the limit of $X_\delta(t)$ be $X(t)$, then $X(t)$ is a continuous-time random process and we have:

$$E[X(t)] = 0$$

$$Var[X(t)] = \lim_{\delta \to 0} h^2 n = \lim_{\delta \to 0} (\sqrt{\alpha\delta})^2 \frac{t}{\delta} = \alpha t$$

- $X(t)$ is called the *Wiener process*. It is used to model *Brownian motion*, the motion of particles suspended in a fluid that move under the rapid and random impact of neighbooring particles.

Wiener Process

- Note that since $\delta = \frac{t}{n}$,

$$X_{\delta}(t) = hS_n = \sqrt{\alpha\delta}S_n = \frac{S_n}{\sqrt{n}}\sqrt{\alpha t}$$

When $\delta \to 0$, $n \to \infty$ and since $\mu_D = 0$, $\sigma_D = 1$, from CLT, we have

$$\frac{S_n}{\sqrt{n}} = \frac{S_n - n\mu_D}{\sigma_D\sqrt{n}} \sim N(0,1)$$

So the distribution of $X(t)$ follows

$$X(t) \sim N(0, \alpha t)$$

i.e.

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}}e^{\frac{-x^2}{2\alpha t}}$$

Wiener Process

- Since *Wiener process* is a limit of random walk, it inherits the properties such as independent and stationary increments. So the joint pdf of $X(t)$ at $t_1, t_2, \cdots, t_k$ $(t_1 < t_2 < \cdots < t_k)$ will be

$$
\begin{aligned}
& f_{X(t_1),X(t_2),\cdots,X(t_k)}(x_1, x_2, \cdots, x_k) \\
&= f_{X(t_1)}(x_1) f_{X(t_2-t_1)}(x_2 - x_1) \cdots f_{X(t_k-t_{k-1})}(x_k - x_{k-1}) \\
&= \frac{\exp\{-\frac{1}{2}[\frac{x_1^2}{\alpha t_1} + \cdots + \frac{(x_k - x_{k-1})^2}{\alpha(t_k - t_{k-1})}]\}}{\sqrt{(2\pi\alpha)^k t_1 (t_2 - t_1) \cdots (t_k - t_{k-1})}}
\end{aligned}
$$

Wiener Process

- mean funtion: $m_{\mathsf{X}}(t) = E[X(t)] = 0$
  auto-covariance: $C_{\mathsf{X}}(t_1, t_2) = \alpha \min(t_1, t_2)$
  Proof:

$$
\begin{aligned}
X_{\square}(t) &= hS_{\mathsf{n}} \\
C_{\mathsf{X}_{\square}}(t_1, t_2) &= h^2 C_{\mathsf{S}}(n_1, n_2) \quad (\text{where } n_1 = \frac{t_1}{\delta}, n_2 = \frac{t_2}{\delta}) \\
&= (\sqrt{\alpha\delta})^2 \min(n_1, n_2)\sigma_{\mathsf{D}}^2 \\
&\qquad (\text{keep in mind that: } \sigma_{\mathsf{D}}^2 = 1) \\
&= \alpha \min(n_1\delta, n_2\delta) = \alpha \min(t_1, t_2)
\end{aligned}
$$

Stationary Random Processes

- $X(t)$ is *stationary* if the joint distribution of any set of samples does not depend on the placement of the time origin.
$$F_{X(t_1),X(t_2),\cdots,X(t_k)}(x_1,x_2,\cdots,x_k)$$
$$= F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_k+\tau)}(x_1,x_2,\cdots,x_k)$$
for all time shift τ, all k, and all choices of $t_1,t_2,\cdots,t_k$.
- $X(t)$ and $Y(t)$ are *joint stationary* if the joint distribution of $X(t_1),X(t_2),\cdots,X(t_k)$ and $Y(t'_1),Y(t'_2),\cdots,Y(t'_j)$ do not depend on the placement of the time origin for all k,j and all choices of $t_1,t_2,\cdots,t_k$ and $t'_1,t'_2,\cdots,t'_j$.

Stationary Random Processes

- First-Order Stationary

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x), \quad \text{for all } t, \tau$$

$$\Rightarrow \quad m_X(t) = E[X(t)] = m, \quad \text{for all } t$$

$$VarX(t) = E[(X(t) - m)^2] = \sigma^2, \quad \text{for all } t$$

- Second-Order Stationary

$$F_{X(t_1)X(t_2)}(x_1, x_2) = F_{X(0)X(t_2-t_1)}(x_1, x_2), \quad \text{for all } t_1, t_2$$

$$\Rightarrow \quad R_X(t_1, t_2) = R_X(t_1 - t_2), \quad \text{for all } t_1, t_2$$

$$C_X(t_1, t_2) = C_X(t_1 - t_2), \quad \text{for all } t_1, t_2$$

The auto-correlation and auto-covariance depend only on the time difference.

Stationary Random Processes

Example:

- An i.i.d random process is stationary.

$$
\begin{aligned}
& F_{X(t_1),X(t_2),\cdots,X(t_k)}(x_1, x_2, \cdots, x_k) \\
&= F_X(x_1)F_X(x_2)\cdots F_X(x_k) \\
&= F_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_k+\tau)}(x_1, x_2, \cdots, x_k)
\end{aligned}
$$

- sum of i.i.d random process $S_n = X_1 + X_2 + \cdots + X_n$
  We know $m_S(n) = nm$ and $Var[S_n] = n\sigma^2$

  $\Rightarrow$ not stationary.

Wide-Sense Stationary (WSS)

- $X(t)$ is WSS if:

$$\begin{aligned} m_{\mathsf{X}}(t) &= m, \quad \text{for all } t \\ C_{\mathsf{X}}(t_1, t_2) &= C_{\mathsf{X}}(t_1 - t_2), \quad \text{for all } t_1, t_2 \end{aligned}$$

  Let $\tau = t_1 - t_2$, then $C_{\mathsf{X}}(t_1, t_2) = C_{\mathsf{X}}(\tau)$.

Wide-Sense Stationary (WSS)

- Example: Let X_n consists of two interleaved sequences of independent r.v.s.
  For n even: $X_n \in \{+1, -1\}$ with $p = \frac{1}{2}$
  For n odd: $X_n \in \{\frac{1}{3}, -3\}$ with $p = \frac{9}{10}$ and $\frac{1}{10}$ resp.
  Obviously, X_n is not stationary, since its pmf varies with n. However,

$$
\begin{aligned}
m_X(n) &= 0 \quad \text{for all } n \\
C_X(i,j) &= \begin{cases} E[X_i]E[X_j] = 0, & i \neq j \\ E[X_i^2] = 1, & i = j \end{cases} \\
&= \delta_{i,j} \\
&\Rightarrow X_n \text{ is WSS.}
\end{aligned}
$$

  So stationary $\Rightarrow$ WSS, WSS $\not\Rightarrow$ stationary.

Autocorrelation of WSS processes

- $R_X(0) = E[X^2(t)]$, for all t.
  $R_X(0)$: average power of the process.
- $R_X(\tau)$ is an even function.
  $R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$
- $R_X(\tau)$ is a measure of the rate of change of a r.p.

$$P\{|X(t+\tau) - X(t)| > \varepsilon\} = P\{(X(t+\tau) - X(t))^2 > \varepsilon^2\}$$

$$\leq \frac{E[(X(t+\tau) - X(t))^2]}{\varepsilon^2} \quad \text{(Markov Inequality)}$$

$$= \frac{2[R_X(0) - R_X(\tau)]}{\varepsilon^2}$$

If $R_X(\tau)$ is flat $\Rightarrow [R_X(0) - R_X(\tau)]$ is small $\Rightarrow$ the probability of having a large change in $X(t)$ in τ seconds is small.

Autocorrelation of WSS processes

- $|R_X(\tau)| \leq R_X(0)$
  Proof: $E[(X(t+\tau) \pm X(t))^2] = 2[R_X(0) \pm R_X(\tau)] \geq 0$
  $$\Rightarrow |R_X(\tau)| \leq R_X(0)$$
- If $R_X(0) = R_X(d)$, then $R_X(t)$ is periodic with period d, and $X(t)$ is *mean square periodic*, i.e.,
  $$E[(X(t+d) - X(t))^2] = 0$$
  Proof: read textbook (pg.360). Use the inequality $E[XY]^2 \leq E[X^2]E[Y^2]$ (from $|\rho| \leq 1$, sec.4.7)
- If $X(t) = m + N(t)$, where $N(t)$ is a zero-mean process s.t. $R_N(\tau) \to 0$, as $\tau \to \infty$, then
  $R_X(\tau) = E[(m+N(t+\tau))(m+N(t))] = m^2 + R_N(t) \to m^2$ as $\tau \to \infty$.

Autocorrelation of WSS processes

- $R_X(\tau)$ can have three types of components: (1) a component that $\to 0$, as $\tau \to \infty$, (2) a periodic component, and (3) a component that due to a non zero mean.
- Example: $R_X(\tau) = e^{\square 2 \square | \square |}$, $R_Y(\tau) = \frac{a^2}{2}\cos 2\pi f_0 \tau$
  If $Z(t) = X(t) + Y(t) + m$ and assume X, Y are independent with zero mean, then
  $$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + m^2$$

WSS Gausian Random Process

- If a Gaussian r.p. is WSS, then it is stationary (Strict Sense Stationary)
  Proof:
  $$f_{\underline{X}}(\underline{x}) = \frac{\exp\{-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})\}}{(2\pi)^{\frac{n}{2}}|K|^{\frac{1}{2}}}$$
  $$\underline{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_n) \end{bmatrix} \quad K = \begin{bmatrix} C_X(t_1,t_1) & \cdots & C_X(t_1,t_n) \\ \vdots & \vdots & \vdots \\ C_X(t_n,t_1) & \cdots & C_X(t_n,t_n) \end{bmatrix}$$
  If $X(t)$ is WSS, then $m_X(t_1) = m_X(t_2) = \cdots = m$, $C_X(t_i,t_j) = C_X(t_i - t_j)$. So $f_{\underline{X}}(\underline{x})$ does not depend on the choice of the time origin $\Rightarrow$ Strict Sense Stationary.

Cyclo Stationary Random Process

- Read on your own.

Continuity of Random Process

- Recall that for $X_1, X_2, \cdots, X_n, \cdots$
  $X_n \to X$ in m.s. (mean square) if $E[(X_n - X)^2] \to 0$, as $n \to \infty$
- Cauchy Criterion
  If $E[(X_n - X_m)^2] \to 0$ as $n \to \infty$ and $m \to \infty$, then $\{X_n\}$ converges in m.s.
- Mean Square Continuity
  The r.p. $X(t)$ is continuous at $t = t_0$ in m.s. if
  $$E[(X(t) - X(t_0))^2] \to 0, \quad \text{as } t \to t_0$$
  We wrtie it as: $\text{l.i.m.}_{t \to t_0} X(t) = X(t_0)$ (limit in the mean)

Continuity of Random Process

- Condition for mean square continuity
  $E[(X(t)-X(t_0))^2] = R_X(t,t)-R_X(t_0,t)-R_X(t,t_0)+R_X(t_0,t_0)$
  If $R_X(t_1,t_2)$ is continuous (both in t_1,t_2), at point (t_0,t_0), then $E[(X(t)-X(t_0))^2] \to 0$. So $X(t)$ is continuous at t_0 in m.s. if $R_X(t_1,t_2)$ is continuous at (t_0,t_0)
- If $X(t)$ is WSS, then:
  $$E[(X(t_0+\tau)-X(t_0))^2] = 2(R_X(0)-R_X(\tau))$$
  So $X(t)$ is continuous at t_0, if $R_X(\tau)$ is continuous at $\tau = 0$

Continuity of Random Process

- If $X(t)$ is continuous at t_0 in m.s., then

$$\lim_{t \to t_0} m_X(t) = m_X(t_0)$$

Proof:

$$\begin{aligned} & Var[X(t) - X(t_0)] \geq 0 \\ \Rightarrow \quad & E[X(t) - X(t_0)]^2 \leq E[(X(t) - X(t_0))^2] \to 0 \\ \Rightarrow \quad & (m_X(t) - m_X(t_0))^2 \to 0 \\ \Rightarrow \quad & m_X(t) \to m_X(t_0) \end{aligned}$$

Continuity of Random Process

- Example: Wiener Process:
$$R_X(t_1, t_2) = \alpha \min(t_1, t_2)$$
$R_X(t_1, t_2)$ is continous at $(t_0, t_0) \Rightarrow X(t)$ is continuous at t_0 in m.s.
- Example: Poisson Process:
$$C_N(t_1, t_2) = \lambda \min(t_1, t_2)$$
$$R_N(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$
$N(t)$ is continuous at t_0 in m.s.
Note that for any sample poisson process, there are infinite number of discontinuities, but $N(t)$ is continuous at any t_0 in m.s.

Mean Square Derivative

- The mean square derivative $X'(t)$ of the r.p. $X(t)$ is defined as:

$$X'(t) = \underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

  provided that

$$\lim_{\varepsilon \to 0} E\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right)^2\right] = 0$$

- The mean square derivative of $X(t)$ at t exists if $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$ exists at point (t, t).
  Proof: read on your own.
- For a Gaussian random process $X(t)$, $X'(t)$ is also Gaussian

Mean Square Derivative

- Mean, cross-correlation, and auto-correlation of $X'(t)$

$$
\begin{aligned}
m_{X'}(t) &= \frac{d}{dt} m_X(t) \\
R_{XX'}(t_1, t_2) &= \frac{\partial}{\partial t_2} R_X(t_1, t_2) \\
R'_X(t_1, t_2) &= \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)
\end{aligned}
$$

- When $X(t)$ is WSS,

$$
\begin{aligned}
m_{X'}(t) &= 0 \\
R_{XX'}(\tau) &= \frac{\partial}{\partial t_2} R_X(t_1 - t_2) = -\frac{d}{d\tau} R_X(\tau) \\
R_{X'}(\tau) &= \frac{\partial}{\partial t_1} \left(\frac{\partial}{\partial t_2} R_X(t_1 - t_2) \right) = -\frac{d^2}{d\tau^2} R_X(\tau)
\end{aligned}
$$

Mean Square Derivative

- Example: Wiener Process

  $$R_X(t_1, t_2) = \alpha \min(t_1, t_2) \Rightarrow \frac{\partial}{\partial t_2} R_X(t_1, t_2) = \alpha u(t_1 - t_2)$$

  $u(\cdot)$ is the step function and is discontinuous at $t_1 = t_2$. If we use the delta function,

  $$R_{X'}(t_1, t_2) = \frac{\partial}{\partial t_1} \alpha u(t_1, t_2) = \alpha \delta(t_1 - t_2)$$

  Note $X'(t)$ is not physically feasible since $E[X'(t)^2] = \alpha\delta(0) = \infty$, i.e., the signal has infinite power. When $t_1 \neq t_2$, $R_{X'}(t_1, t_2) = 0 \Rightarrow X'(t_1), X'(t_2)$ uncorrelated (note $m_{X'}(t) = 0$ for all t) $\Rightarrow$ independent since $X'(t)$ is a Gaussian process.

- $X'(t)$ is the so called White Gaussian Noise.

Mean Square Integrals

- The mean square integral of $X(t)$ form t_0 to t: $Y(t) = \int_{t_0}^{t} X(t')dt'$ exists if the integral $\int_{t_0}^{t}\int_{t_0}^{t} R_X(u,v)dudv$ exists.
- The mean and autocorrelation of $Y(t)$

$$m_Y(t) = \int_{t_0}^{t} m_X(t')dt'$$

$$R_Y(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u,v)dudv$$

Ergodic Theorems

- Time Averages of Random Processes and Ergodic Theorems.

Read on your own.