ENCS 6161

Probability and Random Processes

Fall 2004

X 1st. lecture, Sept. 8, 2004

- Course outline.
- Course Objective.

Applications of Probability & Random Processes:

- Digital Communications.
- Communication Networks.
- Signal Processing
- Reliability

- relative frequency $\longrightarrow$ Axiomatic approach

Assume that we repeat a random experiment with K possible outcomes n times and record the number of times $k=1,2,\cdots,K$ th outcome ~~event~~ occurs as $N_k(n)$ then frequency of occurencen of k is $\frac{N_k(n)}{n}$. We expect that as the number of trials, n, tends to ∞, the ratio $\frac{N_k(n)}{n} \rightarrow P_k$ $k=1,\cdots,N$ where P_k is

the probability of k-th outcome. Note that $P_k \leq 1$.
Note that for two disjoint outcomes k and ℓ, we have $N_{k,\ell}(n) = N_k(n) + N_\ell(n)$ and, therefore,

$$P_{k,\ell} = P_k + P_\ell .$$

Also, we have $\sum_{k=1}^{K} N_k(n) = n$. So,

$$\sum_{k=1}^{K} P_k = 1 .$$

An approach, equivalent to the above, but more abstract is formulated based on the construction of the probabilistic model using a few axioms. This approach is called axiomatic probability theory.

The basic building blocks of the axiomatic probability are: S

1) A Sample Set: A set of outcomes (basic, disjoint possible results of a random experiment)
2) A set of subsets of S each called an event.

3) A probability measure: Assigning a value to each event.

examples of the sample space:

$S = \{H, T\}$ in tossing a coin

$S = \{1, 2, \dots, 6\}$ in throwing a dice.

$S = \{(1,1), (1,2), \dots, (6,6)\}$ in throwing two ~~dice~~ die.

$S = \{2, 3, 4, \dots, 12\}$ the sum of two numbers when throwing two die.

$S = \{x : -V_p \leq x \leq V_p\}$ when sampling a voltage source with peak value V_p.

The sample space can be either <u>finite</u>, <u>countable</u> or <u>uncountable</u>.

<u>Events</u>: Each subset $A \subseteq S$ constitutes an event take $S = \{1, \dots, 6\}$

$A = \{1, 3, 5\}$ is the event of having an odd number on the dice $A = S = \{1, \dots, 6\}$ is the sure event and $A = \Phi = \{\}$ is the nul event.

similarly for $S = \{x : -V_p \leq x \leq V_p\}$

$A = \{x : 0 \leq x \leq V_p\}$ is the event that the sampled value is positive and $A = \{x : -\frac{V_p}{2} \leq x \leq \frac{V_p}{2}\}$ is the event that the voltage is between $-\frac{V_p}{2}$ and $+\frac{V_p}{2}$.

A probability Measure is a <u>set function</u> $P(\cdot)$ with the following properties:

1) $P(A) \geq 0$

2) $P(S) = 1$ equivalently $0 \leq P(A) \leq 1$
(we will show this)

3) if $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

3') if $A_1, A_2, \ldots$ are events such that $A_i \cap A_j = \phi$ for all $i \neq j$ then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P(A_k)$$

Corollary 1: $P(A^c) = 1 - P(A)$

$A^c \cap A = \phi \Rightarrow P(A^c \cup A) = P(A^c) + P(A)$

$P(A^c \cup A) = P(S) = 1$.

So, $P(A^c) = 1 - P(A)$.

Corollary 2: $P(A) \leq 1 \quad \forall A$

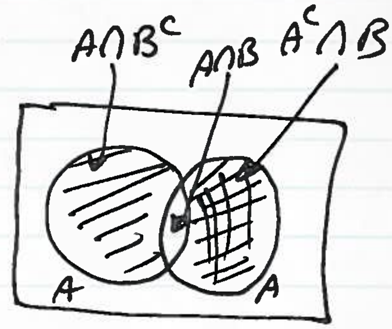
$$P(A) = 1 - P(A^c) \leq 1 \quad \text{since } P(A^c) \geq 0.$$

Corollary 3: $P(\phi) = 0$

$$P(\phi) = 1 - P(\phi^c) = 1 - P(S) = 0$$

Corollary 4: if $A_1, \cdots, A_n$ are mutually exclusive events, i.e., $A_i \cap A_j = \phi \quad \forall i \neq j$

then $\boxed{P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)}$



Corollary 5:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

we have $A \cup B = (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B)$

$$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$$

Similarly:

$$P(A) = P(A \cap B^c) + P(A \cap B)$$
$$P(B) = P(A^c \cap B) + P(A \cap B)$$

so,

$$P(A \cup B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B$$

or $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$

Union bound: ~~[illegible]~~

$$P(A \cup B) \leq P(A) + P(B)$$

Combinatorics:

1) Permutation: Assume that we have n objects. In how many ways can we arrange them? the first one, we can place in in any n locations, the second in any remaining locatio[ns] etc. So $P_n = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$

2) ~~Combination~~ selection with order: In how many ways we can select k objects from $n \geq k$ objects? For the 1st object we have n choices, for the second $n-1$, ... for the kth $n-k+1$

So $P_k^n = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$

3) Combination (selection without order): In how many ways we can select k objects from $n \geq k$ objects (here order is not our concer[n]) Assume this number is C_k^n. That is, there are C_k^n ways in which k objects can be selected from

a set of n objects. Each one of these C_k^n sets can be arranged in $k!$ ways so:

$k!\, C_k^n$ is the total number of ways k objects from a set of size n can be formed.

Or,

$$k!\, C_k^n = P_k^n = \frac{n!}{(n-k)!}$$

So

$$C_k^n = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

This is called n-choose-k.

<u>Example</u>: Find the number of ways one can select 2 objects from $A = \{1, 2, 3, 4, 5\}$

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

DTMF Signalling

Conditional Probability

Take two events A and B

$$P[A \cap B] = P[A]\, P[B|A]$$

i.e., probability of A and B occuring is the probability of A occuring times the probability of B occuring given A has occured.

$P(B|A)$ is called the conditional probability of B given A:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Similarly:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

or

$$P(A \cap B) = P(B)\, P(A|B) = P(A)\, P(B|A)$$

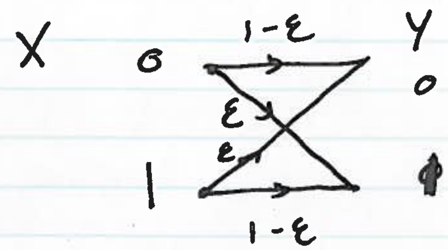
so,

$$P(A|B) = \frac{P(A)\, P(B|A)}{P(B)}$$

This is called the <u>Bayes Rule</u>.

Example:

Consider a communication channel transmitting binary data with probability of error ε. Assume that bits at its input have $P(0) = p$ and $P(1) = 1 - p$.



We have:

$$P(X=0, Y=0) = P(0,0) = p(1-\varepsilon)$$

$$P(X=0, Y=1) = p\varepsilon$$

$$P(X=1, Y=0) = (1-p)\varepsilon$$

$$P(X=1, Y=1) = (1-p)(1-\varepsilon)$$

$$P(X=0|Y=0) = \frac{P(X=0)\,P(Y=0|X=0)}{P(Y=0)}$$

$$= \frac{p(1-\varepsilon)}{p(1-\varepsilon) + (1-p)\varepsilon}$$

$$P(X=1|Y=0) = \frac{P(X=1)\,P(Y=0|X=1)}{P(Y=0)} = \frac{(1-p)\varepsilon}{p(1-\varepsilon)+(1-p)\varepsilon}$$

Upon observing Y, we form the a posterior probabilities $P(X=0|Y=y)$ and $P(X=1|Y=y)$ for $Y=y\in\{0,1\}$ and decide in favour of the one being more probable (MAP detection

For $Y=0$ for example, we ~~but~~ decide $X=0$ if

$$P(X=0|Y=0) > P(X=1|Y=0)$$

or

$$P(X=0)P(Y=0|X=0) > P(X=1)P(Y=0|X=1)$$

or $\quad p(1-\varepsilon) > (1-p)\varepsilon \quad$ else, we decide $X=1$

$(p>\varepsilon \rightarrow X=0)$

For $Y=1$, we decide $X=0$ if

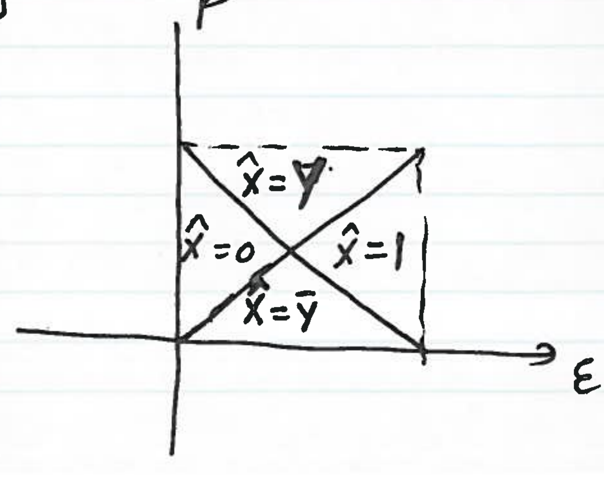
$$P(X=0)P(Y=1|X=0) > P(X=1)P(Y=1|X=1)$$

or

$p\varepsilon > (1-p)(1-\varepsilon) \quad$ else, we decide $X=1$

$[(p>1-\varepsilon) \rightarrow X=0]$

or $1-p<\varepsilon$



Example (Bayes Rule)

Assume that you are given the following information:

- 30% of the adults in a given city are smokers.
- Smokers die of lung cancer at a rate of 1 in 100,000.
- ~~no~~ fatality amongst non-smokers (of lung cancer) is 1 in 100 millions.

A person dies of lung cancer. You are asked to decide whether he has been a smoker or not.

a) What is your answer?

b) What is the probability that you are wrong?

$P(S) = 0.3 \qquad P(\bar{S}) = 0.7$

$P(L.C.|S) = 10^{-5} \qquad P(L.C|\bar{S}) = 10^{-8}$

$$P(S|L.C.) = \frac{P(S)\,P(L.C.|S)}{P(L.C.)} = \frac{0.3\times10^{-5}}{0.3\times10^{-5}+0.7\times10^{-8}}$$

$P(S|L.C.) = 0.9977$ and $P(\bar{S}|L.C.) = 0.0023$

a) He was smoker. b) $P_e = 2.3\times10^{-3}$

independence

two events A and B are said to be independent if knowing that B has occurred does not alter the probability of A happening

i.e.,

$$P(A|B) = P(A)$$

For two independent events A and B, we have

$$P(A \cap B) = P(A)P(B|A) = P(A)P(B)$$

For independent events $A_1, A_2, \cdots A_n$

$$P(A_1 \cap A_2 \cdots \cap A_n) = \prod_{i=1}^{n} P(A_n)$$

In general, when events are not necessarily independent

$$P(A_1 \cdots A_n) = P(A_1)P(A_2|A_1)\,P(A_3|A_1A_2)\cdots P(A_n|A_1 \cdots$$

$$= \prod_{i=1}^{n} P(A_i|A_1 \cdots A_{i-1})$$