X Lecture 10, Nov. 10, 04

- Variance, mean, and autocovariance of ~~random walk~~ sum of i.i.d. processes.

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$m_S(n) = E[S_n] = nE[X] = nm$$

$$Var[S_n] = n\,Var(X) = n\sigma^2$$

Autocovarian is:

$$C_S(n,k) = E[(S_n - E(S_n))(S_k - E(S_k))]$$

$$= E[(S_n - nm)(S_k - km)]$$

$$= E\left[\sum_{i=1}^{n}(X_i - m)\sum_{j=1}^{k}(X_j - m)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E[(X_i - m)(X_j - m)]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} C_X(i,j)$$

$$C_X(i,j) = \sigma^2 \delta_{i,j}$$

So

$$C_S(n,k) = \sum_{i=1}^{\min(n,k)} C_X(i,i) = \min(k,n)\sigma^2$$

Example: Find mean, variance and autocovariance of random walk.

$E[S_n] = nm = n(2p-1)$

$Var(S_n) = n\sigma^2 = 4np(1-p)$

$C_S(n,k) = \min(n,k)\, 4p(1-p)$

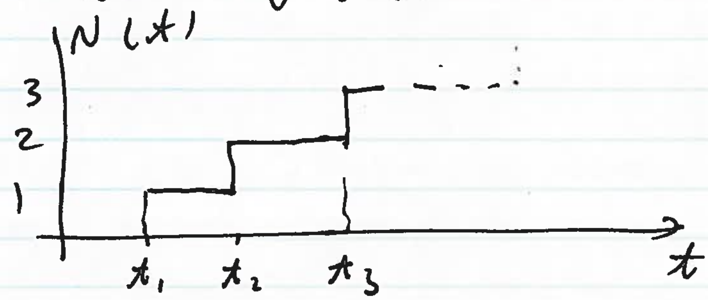
Continuous Random Processes

Poisson Process

examples: arrival of print jobs at a printer, patients at a physician's office, packets of data at a router port, etc.

$N(t) \triangleq$ # of arrivals in $[0,t]$

$\lambda \triangleq$ rate of arrival.



let's divide $[0, t]$ into n (n is large) segments of each $\delta = \frac{t}{n}$ seconds wide.

Assume that:

1) The probability of more than one arrival (eve... in δ is negligibly small. We can make this to be true by increasing n.

2) Whether or not an event (arrival) occurs in a subinterval is independent of arrivals in other subintervals.

So:

1) The arrivals in each sub-interval are Bernouli distributed.

2) The Bernouli arrivals are independent.

The rate of arrival is λ so in t seconds we have λt arrivals on the average. On the other hand, if we assume that ~~the~~ probability of arrival in δ is p (the parameter of Bernouli variabl...

then $\lambda t = np$

since ~~the~~ average number of arrivals is np.

So, we have a <u>binomial</u> distribution for the sum process that tends to a Poisson as $n \to \infty$ (i.e., $\delta \to 0$) and $p \to 0$ while $np = \lambda t$ and we have

$$P[N(t)=k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad k = 0, 1, 2, \ldots$$

Inter-event time

Let's again divide $[0, t]$ into n segments (sub-intervals) with length $\delta = \frac{t}{n}$. Denote the interevent time by T.

Probabability that $T > t$ is the probability that no event occurs in t seconds.

$$P[T > t] = P[\text{no arrivals in } t \text{ seconds}]$$

$$= (1-p)^n$$

$$= \left(1 - \frac{\lambda t}{n}\right)^n \xrightarrow[n \to \infty]{} e^{-\lambda t}$$

$$F_T(t) = P[T \leq t] = 1 - e^{-\lambda t}$$

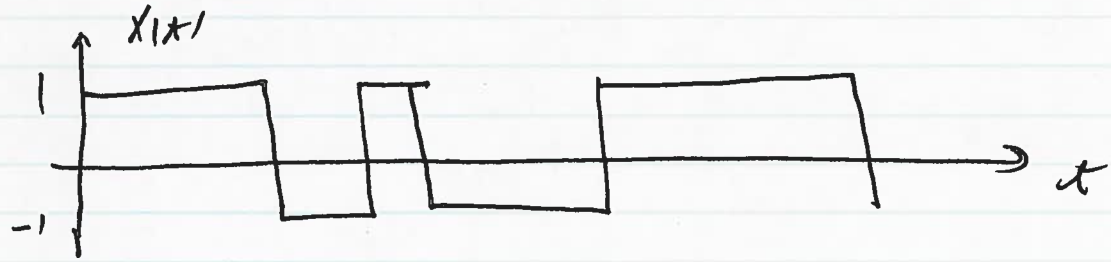
and

$$f_T(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$

So, the inter-arrival times (or MTTF) is exponential with mean $1/\lambda$. mean-time-to-failure

Random Telegraph Signal

Let $X(t) = \pm 1$ and assume that $X(0) = \pm 1$ with equal probability, $\left(P[X(0)=1] = \frac{1}{2}\right)$. Also assume that $X(t)$ changes polarity with each occurence of an event in a Poisson Process. A realization of this process



$$P[X(t) = \pm 1] = P[X(0)=1]\, P[X(t) = \pm 1 \mid X(0) = 1]$$
$$+ P[X(0) = -1]\, P[X(t) = \pm 1 \mid X(0) = -1]$$

$X(t)$ would have the same polarity as $X(0)$ if the number of arrivals (Poisson events) is even.

$$P[X(t) = \pm 1 \mid X(0) = \pm 1] = P[N(t) = \text{even}]$$
$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t} = \frac{1}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) e^{-\alpha t}$$
$$= \frac{1}{2}\left(1 + e^{-2\alpha t}\right)$$

$$P[X(t) = \pm 1 \mid X(0) = \mp 1] = P[N(t) = \text{odd}]$$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$

$$= e^{-\alpha t} \cdot \frac{1}{2}(e^{\alpha t} - e^{-\alpha t})$$

$$= \frac{1}{2}(1 - e^{-2\alpha t})$$

$$P[X(t) = 1] = \frac{1}{2} \cdot \frac{1}{2}(1 + e^{-2\alpha t}) + \frac{1}{2} \cdot \frac{1}{2}(1 - e^{-2\alpha t}) = \frac{1}{2}$$

$$P[X(t) = -1] = \frac{1}{2}$$

$$m_X(t) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$$

$$Var(X(t)) = E[X^2(t)] = (1)^2 P(X(t) = 1) + (-1)^2 P(X(t) = -1)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$C_X(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= 1 \times P[X(t_1) = X(t_2)] + (-1) P[X(t_1) \neq X(t_2)]$$

$$= 1 \times \frac{1}{2}\{1 + e^{-2\alpha|t_2 - t_1|}\} - \frac{1}{2}\{1 - e^{-2\alpha|t_2 - t_1|}\}$$

$$\boxed{= e^{-2\alpha|t_2 - t_1|}}$$

Wiener Process

Take a symmetric random walk ($p = \frac{1}{2}$) every δ seconds with increments $\pm h$. Then, we have

$$X_\delta(t) = h(D_1 + D_2 + \cdots + D_n) = h S_n$$

where D_i is a ~~process~~ variable taking ± 1 with equal probability.

$$E[X_\delta(t)] = h E[S_n] = 0$$

$$Var[X_\delta(t)] = h^2 n \; Var(D_n) = h^2 n$$

since

$$Var(D_n) = 4p(1-p)\Big|_{p=1/2} = 1$$

Let $\delta \to 0$ and $h \to 0$ with $h = \sqrt{\alpha \delta}$

then $X_\delta(t) \to X(t)$ where

$$E[X(t)] = 0$$

$$Var(X(t)) = \left(\sqrt{\alpha\delta}\right)^2 \left(\frac{t}{\delta}\right) = \alpha t$$

This continuous-time process is called the Wiener Process and is used to model the Brownian motion.

Note that while the mean is zero the variance is linear with time. $\rightarrow$ Defusion

As $\delta \to 0$, $n = [\frac{t}{\delta}] \to \infty$.

So $X(t)$ is the sum of an infinite number of i.i.d. random variables. Due to central limit theorem

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t}$$

Wiener process inherits the properties of independent and stationary increments from the random walk process. So,

$$f_{X(t_1)\cdots X(t_n)}(x_1, \cdots x_n) = f_{X(t_1)}(x_1) f_{X(t_2-t_1)}(x_2-x_1) \cdots f_{X(t_k-t_{k-1})}(x_k-x_{k-1})$$

$$= \frac{\exp\left[-\frac{1}{2}\left[\frac{x_1^2}{2} + \frac{(x_2-x_1)^2}{\alpha(t_2-t_1)} + \cdots + \frac{(x_k-x_{k-1})^2}{\alpha(t_k-t_{k-1})}\right]\right]}{\sqrt{(2\pi\alpha)^k t_1(t_2-t_1)\cdots(t_k-t_{k-1})}}$$

The autocovariance of $X(t)$ is given as:

$$C_X(t_1, t_2) = \alpha \min(t_1, t_2)$$