X Lecture 11, Nov. 17, 04

Stationary Random Processes

$X(t)$ is stationary if the joint distribution of any set of samples of it does not depend on the placement of time origin.

Two processes are called jointly stationary if the joint cdf of $X(t_1) \cdots X(t_k)$ and $Y(t_1'), \cdots Y(t_j')$ do not depend on the placement of the time origin for any k, j and choice of $\{t_j\}$, $\{t_k'\}$.

First order stationarity

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \quad \text{all } t, \tau$$

So,

$$m_X(t) = E[X(t)] = m \quad \text{all } t$$

$$Var(X(t)) = E[(X(t)-m)^2] = \sigma^2 \quad \text{all } t$$

The second order stationarity

$$F_{X(t_1), X(t_2)}(x_1, x_2) = F_{X(0), X(t_2 - t_1)}(x_1, x_2) \quad \text{all } t_1, t_2$$

So:

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \quad \text{all } t_1, t_2$$

$$C_X(t_1, t_2) = C_X(t_2 - t_1) \quad \text{all } t_1, t_2 .$$

<u>Wide sense stationary</u> random Processes:

$$m_X(t) = m \quad \text{all } t$$

$$C_X(t_1, t_2) = C_X(t_1 - t_2) \quad \text{all } t_1, t_2 .$$

<u>Example</u>: The sum of i.i.d. random processes $S_n = X_1 + X_2 + \cdots + X_n$

$$m_S(n) = nm \quad \text{and} \quad Var(S_n) = n\sigma^2$$

The mean and variance depend on time (n).

So the process is not stationary.

Ex. Let X_n consist of two interleaved independent random sequences.

For n even $X_n \in \{+1, -1\}$ with $p = \frac{1}{2}$.

For n odd $X_n \in \{\frac{1}{3}, -3\}$ with $p = 9/10$ and $\frac{1}{10}$, resp.

It is clear that the process is not stationary since its pmf varies with n. However,

$$m_X(n) = 0 \quad \forall n$$

and

$$C_X(i,j) = \begin{cases} E[X_i]E[X_j] = 0 & i \neq j \\ E[X_i^2] = 1 & i = j \end{cases} = \delta_{i,j}$$

So, X_n is WSS.

Properties of <u>autocorrelation function</u> of WSS processes

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

1) $R_X(0) = E[X^2(t)]$ all t

i.e., $R_X(\tau)$ at $\tau = 0$ gives the <u>average energy</u> of the signal.

2) $R_X(\tau)$ is an even function

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau)$$

3) $R_X(\tau)$ is a measure of the rate of change of a random process.

$$P[|X(t+\tau)-X(t)|>\varepsilon] = P[(X(t+\tau)-X(t))^2>\varepsilon^2]$$

$$\leq \frac{E[(X(t+\tau)-X(t))^2]}{\varepsilon^2} = \frac{2[R_X(0)-R_X(\tau)]}{\varepsilon^2}$$

using Markov inequality

The above inequality indicates that if $R_X(\tau)$ is flat, i.e., $[R_X(0)-R_X(\tau)]$ is small, then probability of having a large change in $X(t)$ in τ seconds is small.

4) $R_X(\tau) \leq R_X(0)$

we have use $E[XY]^2 \leq E[X^2]E[Y^2]$

Proof:

$$R_X^2(\tau) = E[X(t+\tau)X(t)]^2 \leq E[X^2(t+\tau)]E[X^2(t)]$$
$$= R_X^2(0)$$

So

$$R_X(\tau) \leq R_X(0)$$

5) If $R_X(0) = R_X(d)$ then $R_X(\tau)$ is periodic with period d and $X(t)$ is <u>mean square periodic</u>, i.e., $E[(X(t+d) - X(t))^2] = 0$

<u>Proof</u>: use the inequality:

$$E[XY]^2 \leq E[X^2]E[Y^2]$$

to

$$\underbrace{X(t+\tau+d) - X(t+\tau)}_{Y} \quad \text{and} \quad \underbrace{X(t)}_{X}$$

then:

$$E[(X(t+\tau+d) - X(t+\tau))X(t)]^2$$

$$\leq E[(X(t+\tau+d) - X(t+\tau))^2]E[X^2(t)]$$

This means:

$$[R_X(\tau+d) - R_X(\tau)]^2 \leq [2R_X(0) - R_X(d)]R_X(0)$$

So:

$$R_X(0) = R_X(d) \Rightarrow R_X(\tau+d) = R_X(\tau) \quad \text{all } \tau.$$

Also

$$E[(X(t+d) - X(t))^2] = 2[R_X(0) - R_X(d)] = 0$$

So $X(t)$ is <u>mean square periodic</u>.

6) if $X(t) = m + N(t)$ where $N(t)$ is a zero-mean process such that $R_N(\tau) \to 0$ as $\tau \to \infty$, then

$$R_X(\tau) = E[(m+N(t+\tau))(m+N(t))] = m^2 + 2m\,\cancel{E[N(t)]}^{0}$$
$$+ R_N(\tau) = m^2 + R_N(\tau) \longrightarrow m^2 \quad \text{as } \tau \to \infty$$

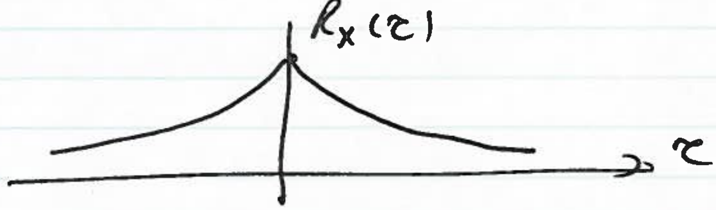
In summary $R_X(\tau)$ approaches the square of the mean of the $X(t)$ as $\tau \to \infty$.

$R_X(\tau)$ can in general have three parts:

1) One part that approaches zero as $\tau \to \infty$
2) A periodic part.
3) A part due to non-zero mean.

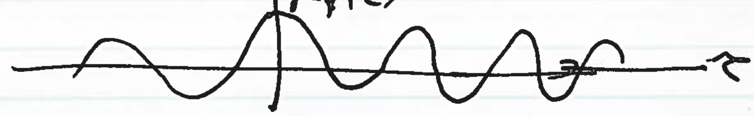
Examples: 1) A random telegraph signal has

$$R_X(\tau) = e^{-2\alpha|\tau|} \quad \forall \tau$$



2) $Y(t) = a\cos(2\pi f_0 t + \theta)$ where θ is uniform random $\in [0, 2\pi]$

has $R_Y(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau)$

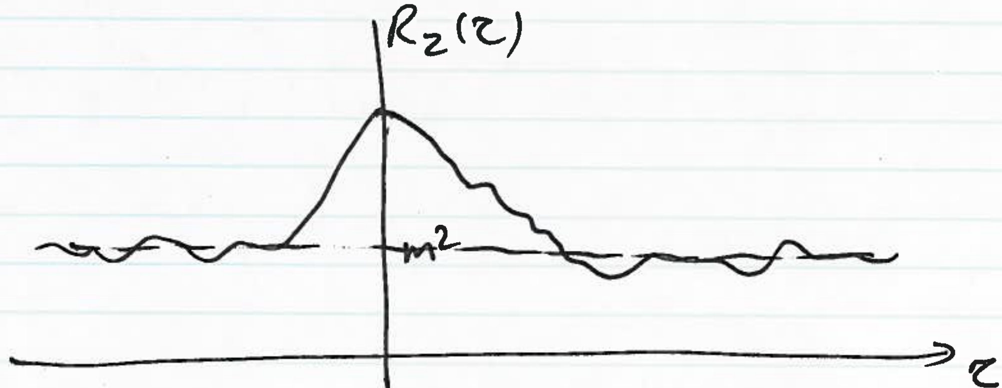


3) $Z(t) = X(t) + Y(t) + m$

where $X(t)$ is a random telegraph signal, $Y(t)$ is sinusoid with random phase and m is a constant has

$$R_Z(\tau) = E[(X(t+\tau) + Y(t+\tau) + m)(X(t) + Y(t) + m)]$$

$$= R_X(\tau) + R_Y(\tau) + m^2$$



~~WSS Gaussian random processes~~

~~recall that~~

$$\cancel{f_{X(t_1)\cdots X(t_n)}(x_1, \ldots x_n) = \frac{\exp\left[-\frac{1}{2}\left\{\frac{x_1^2}{\alpha t_1} + \frac{(x_2 - x_1)^2}{\alpha(t_2 - t_1)} + \cdots \frac{(x_n - x_{n-1})^2}{\alpha(t_n - t_{n-1})}\right\}\right]}{\sqrt{(2\pi\alpha)^n t_1 (t_2 - t_1)(t_3 - t_2) \cdots (t_n - t_{n-1})}}}$$

WSS Gaussian Random Processes:

If a Gaussian random process is wide sense stationary then it is stationary (strict sense stationary).

The reason is that

$$f_{X_1,\cdots X_n}(x_1,\cdots x_n) = \frac{\exp[-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})]}{(2\pi)^{n/2}|K|^{1/2}}$$

where

$$\underline{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_n) \end{bmatrix} \text{ and } K = \begin{bmatrix} C_X(t_1,t_2) & C_X(t_1,t_2) & \cdots & C_X(t_1,t_n) \\ \vdots & & & \\ C_X(t_n,t_1) & \cdots & & C_X(t_n,t_n) \end{bmatrix}$$

if $\underline{X}$ is WSS then $m_X(t_1) = m_X(t_2) = \cdots = m$

and

$$C_X(t_i,t_j) = C_X(t_i - t_j)$$

So, ~~the~~ $f_{\underline{X}}(\underline{x})$ does not depend on the choice of the time origin.

Cyclostationary random processes

A process is <u>cyclostationary</u> if the pdf of (or CDF of) any set of its samples is invariant under a shift of origin by an integer multiple of a period T, i.e.,

$$F_{X(t_1),X(t_2)\cdots X(t_n)}(x_1,\cdots x_n) = F_{X(t_1+mT),X(t_2+mT),\cdots X(t_n+mT)}(x_1,x_2,\cdots x_n)$$

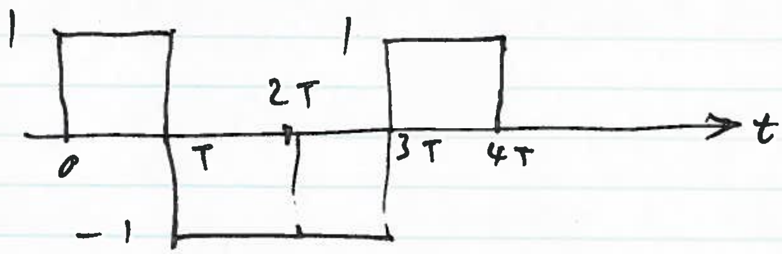
A process is <u>wide sense cyclostationary</u> if

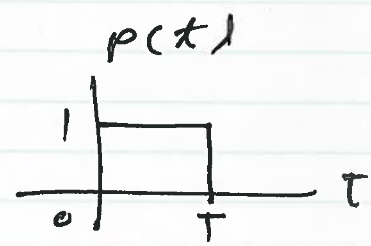
$$m_X(t+mT) = m_X(t)$$

and

$$C_X(t_1+mT, t_2+mT) = C_X(t_1,t_2)$$

<u>Example</u>: A modem transmitting i.i.d. equiprobable binary (± 1) every T seconds.





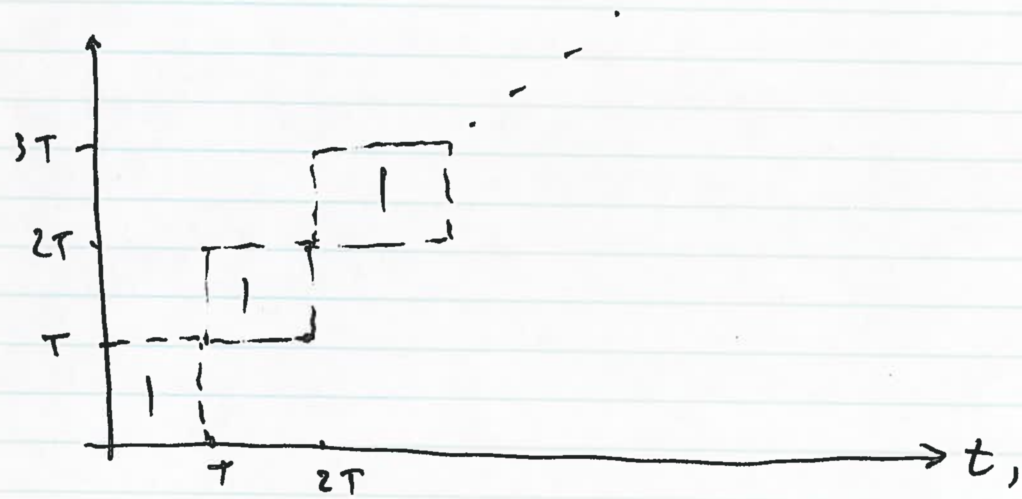
$$X(t) = \sum_{n=-\infty}^{\infty} A_n\, p(t-nT) \quad \text{where}$$

and $A_n \in \{-1, +1\}$

$$m_X(t) = E\left[\sum_{n=-\infty}^{\infty} A_n p(t-nT)\right] = \sum_{n=-\infty}^{\infty} E[A_n]\, p(t-nT) = 0$$

$$C_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \begin{cases} 1 & nT \le t_1 \le (n+1)T \text{ and } nT \le t_2 \le (n+1)T \\ 0 & \text{otherwise.} \end{cases}$$



Note that $C_X(t_1 + mT;\, t_2 + mT) = C_X(t_1, t_2)$

so, it is Wide sense cyclostationary.

- Continuity, Derivatives & integrals of random processes

Mean Square Convergence

We say that a sequence of random variables $X_1(\omega), X_2(\omega), \ldots X_n(\omega), \ldots$ converges to X in the mean square sense if

$$E[(X_n(\omega) - X(\omega))^2] \to 0 \quad \text{as } n \to \infty$$

we denote this as

$$X(\omega) = \text{l.i.m. } X_n(\omega) \quad (\text{limit in mean})$$

When the random variable $X(\omega)$ is not known or is not of interest and we are only interested in knowing whether $X_n(\omega)$ converges or not we use the Cauchy criterion

Cauchy Criterion: The sequence of r.v.'s $\{X_n(\omega)\}$ converges in the mean square sense if:

$$E[(X_n(\omega) - X_m(\omega))^2] \to 0 \quad \text{as } n \to \infty \text{ and } m \to \infty$$

Continuity

For a deterministic function $X(t)$
we say that it is continuous at $t = t_0$ if
given $\varepsilon > 0 \quad \exists \delta > 0 \ni$

$$|t - t_0| < \delta \Rightarrow |X(t) - X(t_0)| < \varepsilon$$

For a random process, each <u>sample function</u>
$X(t, \omega)$ ~~in a~~ can be considered a deterministic
function and we can check for its continuity.
Say, $X(t, \omega)$ is continuous at $t = t_0$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0 \ni |t - t_0| < \delta \Rightarrow |X(t, \omega) - X(t_0, \omega$

Then we write:

$$\lim_{t \to t_0} X(t, \omega) = X(t_0, \omega)$$

except for few cases, e.g., a sinusoid with
random phase, it is difficult to check continuity
for each individual sample function.

Mean Square Continuity:

The random process $X(t)$ is continuous at $t = t_0$
in the mean square sense if

$$E[(X(t) - X(t_0))^2] \to 0 \quad \text{as } t \to t_0$$

we write this as

$$X(t_0) = \text{l.i.m. } X(t) \qquad \text{(limit in the mean)}$$

Condition for Mean Square Continuity

$$E[(X(t) - X(t_0))^2] = E[X^2(t)] - E[X(t)X(t_0)] - E[X(t_0)X(t)] + E[X(t_0)X(t_0)]$$

$$= R_X(t, t) - R_X(t, t_0) - R_X(t_0, t) + R_X(t_0, t_0)$$

if $R_X(t_1, t_2)$ is continuous (both in t_1 and t_2) at point (t_0, t_0) then

$$E[(X(t) - X(t_0))^2] = 0$$

So, $X(t)$ is continuous in the mean square sense at (t_0) if $R_X(t_1, t_2)$ is continuous at (t_0, t_0).

Note that the continuity in the mean square sense does not imply that every sample function of $X(t)$ ~~converges~~ is continuous at t_0. It, however, ~~says that~~ implies that except a number of sample functions with zero probability the rest are continuous.

If a random process $X(t)$ ~~[illegible]~~ is continuous in the mean then $\lim_{t \to t_0} m_X(t) = m_X(t_0)$

The reason is that a discontinuity in $m_X(t)$ requires a set of sample functions with nonzero probability not being continuous at t_0.

To see this note that

$$Var[X(t) - X(t_0)] \geq 0$$

so

$$E[(X(t) - X(t_0))^2] - E[X(t) - X(t_0)]^2 \geq 0$$

so:

$$E[(X(t) - X(t_0))^2] \geq (m_X(t) - m_X(t_0))^2$$

if $X(t)$ is mean square continuous then

$$E[(X(t) - X(t_0))^2] \to 0$$

so $(m_X(t) - m_X(t_0)) \to 0$

or $m_X(t) \to m_X(t_0)$ as $t \to t_0$

or

$$\lim_{t \to t_0} E[X(t)] = E[\underset{t \to t_0}{\mathrm{l.i.m.}}\, X(t)]$$

If the random process $X(t)$ is WSS then

$$E[(X(t_0+\tau)-X(t_0))^2] = 2(R_X(0)-R_X(\tau))$$

So, $X(t)$ is continuous at t_0 if $R_X(\tau)$ is continuous at $\tau = 0$.

Example

1) Wiener Process

$$R_X(t_1,t_2) = \alpha \min(t_1,t_2)$$

So, to see if

$R_X(t_1,t_2)$ is continuous at (t_0,t_0):

$$|R_X(t_0+\varepsilon_1, t_2+\varepsilon_2) - R_X(t_0,t_0)| = \alpha|\min(t_0+\varepsilon_1, t_0+\varepsilon_2) - \varepsilon_1$$

$$= \varepsilon_1 \text{ or } \varepsilon_2 \leq \max(\varepsilon_1,\varepsilon_2)$$

So $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, $|R_X(t_0+\varepsilon_1, t_2+\varepsilon_2) - R_X(t_0,t_0)| \to 0$

So $R_X(t_1,t_2)$ is continuous at (t_0,t_0) so the Wiener process is continuous in the mean square sense.

~~For~~ 2) Poisson Process $N(t)$

$$R_N(t_1,t_2) = \lambda \min(t_1,t_2)$$

Similarly Poisson Process is continuous in the mean square sense. However, while ~~most~~ of the sample functions of a Wiener process are continuous, ~~while~~ every sample function of the Poisson process has an infinite number of discontinuities.

~~Lecture~~ 12, Nov. 24, 04

Mean Square Derivative

The random proces $X(t)$ has mean square derivative $X'(t)$ at t defined as,

$$X'(t) = \underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

provideded that

$$\lim_{\varepsilon \to 0} \left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t) \right)^2 \right] = 0$$

The mean square derivative of $X(t)$ at t exists if $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$ exists at point (t, t).

Proof: We use Cauchy criterion, i.e., we show that

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} - \frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right)^2 \right] \Rightarrow 0$$

$$= E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} \right)^2 \right] + E\left[\left(\frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right)^2 \right]$$

$$- 2E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} \right) \left(\frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right) \right]$$