sample function of the Poisson process has an infinite number of discontinuities.

~~Lecture~~ 12, Nov. 24, 04

Mean Square Derivative

The random proces $X(t)$ has mean square derivative $X'(t)$ at t defined as,

$$X'(t) = \underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

provideded that

$$\lim_{\varepsilon \to 0} \left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t) \right)^2 \right] = 0$$

The mean square derivative of $X(t)$ at t exists if $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$ exists at point (t, t).

Proof: We use Cauchy criterion, i.e., we show that

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} - \frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right)^2 \right] \Rightarrow 0$$

$$= E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} \right)^2 \right] + E\left[\left(\frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right)^2 \right]$$

$$- 2E\left[\left(\frac{X(t+\varepsilon_1) - X(t)}{\varepsilon_1} \right) \left(\frac{X(t+\varepsilon_2) - X(t)}{\varepsilon_2} \right) \right]$$

The first term:

$$E\left[\left(\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}\right)^2\right] = \frac{1}{\varepsilon_1}\left[\frac{R_X(t+\varepsilon_1, t+\varepsilon_1)-R_X(t, t+\varepsilon_1)}{\varepsilon_1} - \frac{R_X(t+\varepsilon_1, t)-R_X(t,t)}{\varepsilon_1}\right]$$

as $\varepsilon_1 \to 0$

$$E\left[\left(\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}\right)^2\right] \to \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)$$

similarly

as $\varepsilon_2 \to 0$

$$E\left[\left(\frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right)^2\right] \to \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)$$

similarly as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$

$$-2E\left[\left(\frac{X(t+\varepsilon_1)-X(t)}{\varepsilon_1}\right)\left(\frac{X(t+\varepsilon_2)-X(t)}{\varepsilon_2}\right)\right]$$

$$= -\frac{2}{\varepsilon_1}\left[\frac{R_X(t+\varepsilon_1, t+\varepsilon_2)-R_X(t+\varepsilon_1, t)}{\varepsilon_2} - \frac{R_X(t, t+\varepsilon_2)-R_X(t,t)}{\varepsilon_2}\right]$$

tends to $-2\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)$

If $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t,t)$ exists then sum of the three terms would be zero.

For The case where $X(t)$ is WSS:

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1 - t_2)$$

$$= \frac{\partial}{\partial t_1}\left[\frac{\partial}{\partial t_2} R_X(t_1 - t_2)\right]$$

$$= \frac{\partial}{\partial t_1}\left[-\frac{\partial}{\partial \tau} R_X(t_1 - t_2)\right] = \boxed{-\frac{d^2}{d\tau^2} R_X(\tau)}.$$

For a Gaussian random process $X(t)$, the derivative $X'(t)$ is also Gaussian.

Mean, Cross-Correlation and auto correlation of the derivative:

$$E[X'(t)] = E\left[\underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon \to 0} E\left[\frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon \to 0} \frac{m_X(t+\varepsilon) - m_X(t)}{\varepsilon} = \frac{d}{dt} m_X(t)$$

Cross-Correlation between $X(t)$ and $X'(t)$

$$R_{X,X'}(t_1,t_2) = E\left[X(t_1)\ \underset{\varepsilon\to 0}{l.i.m.}\ \frac{X(t_2+\varepsilon)-X(t_2)}{\varepsilon}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{R_X(t_1,t_2+\varepsilon)-R_X(t_1,t_2)}{\varepsilon}$$

$$\boxed{= \frac{\partial}{\partial t_2} R_X(t_1,t_2)}$$

Autocorrelation of $X'(t)$

$$R_{X'}(t_1,t_2) = E\left[\underset{\varepsilon\to 0}{l.i.m}\left\{\frac{X(t_1+\varepsilon)-X(t_1)}{\varepsilon}\right\}X'(t_2)\right]$$

$$= \lim_{\varepsilon\to 0}\frac{R_{X,X'}(t_1+\varepsilon,t_2)-R_{X,X'}(t_1,t_2)}{\varepsilon}$$

$$= \frac{\partial}{\partial t_1} R_{X,X'}(t_1,t_2)$$

$$\boxed{= \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1,t_2)}$$

When $X(t)$ is WSS

$$R_{X,X'}(\tau) = \frac{\partial}{\partial t_2} R_X(t_1-t_2) = -\frac{\partial}{\partial \tau} R_X(\tau)$$

and

$$R_{X'}(\tau) = \frac{\partial}{\partial t_1}\left\{\frac{\partial}{\partial t_2} R_X(t_1-t_2)\right\} = \frac{\partial}{\partial t_1}\left\{-\frac{\partial}{\partial \tau} R_X(t_1-t_2)\right\}$$

$$\boxed{= -\frac{d^2}{dt^2} R_X(\tau)}$$

Example: Wiener process

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2) = \begin{cases} \alpha t_2 & t_2 < t_1 \\ \alpha t_1 & t_2 \geq t_1 \end{cases}$$

$$\frac{\partial}{\partial t_2} R_X(t_1, t_2) = \begin{cases} \alpha & t_2 < t_1 \\ 0 & t_2 \geq t_1 \end{cases} = \alpha u(t_1 - t_2)$$

$u(t_1 - t_2)$ is discontinuous at $t_1 = t_2$ so, derivative w.r.t. t_1 (strictly speaking) does not exist. To capture the derivative of a step function at its point of discontinuity, one may use the delta "function" to get

$$\boxed{R_{X'}(t_1, t_2) = \frac{\partial}{\partial t_1} \alpha u(t_1 - t_2) = \alpha \delta(t_1 - t_2)}$$

Note that this is not a physically feasible process as $E[X'(t)^2] = \alpha \delta(0) = \infty$, i.e., the signal has infinite power.

Also note that for any $t_1 \neq t_2$, $R_{X'}(t_1, t_2) = 0$ that means that no matter how close two points in time we take samples of $X'(t)$ at these two points are uncorrelated. This process is the so called ~~AWGN~~ White Gaussian Noise.

Mean Square Integrals

The ~~m~~ integral of a random process $X(t)$, exists (in the mean square sense) if the integral

$$\int_{t_0}^{t}\int_{t_0}^{t} R_X(u,v)\,du\,dv$$

exists.

The mean and variance of $Y(t)=\int_{t_0}^{t} X(t')\,dt'$ are:

$$m_Y(t) = E\left[\int_{t_0}^{t} X(t')\,dt'\right] = \int_{t_0}^{t} E[X(t')]\,dt'$$

$$= \int_{t_0}^{t} m_X(t')\,dt'$$

and

$$R_Y(t_1,t_2) = E\left[\int_{t_0}^{t_1} X(u)\,du \int_{t_0}^{t_2} X(v)\,dv\right]$$

$$= \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u,v)\,du\,dv$$

Reading:

Pages 373 to 378

mean Square integrals

and

Response of linear systems to Random inputs.

Ergodic Theorems

Ergodic Theorems give the condition under which observations in time coincide with the expected values of the observables.

For example, we can find the time average of a single realization of random process $X(t)$, say $X(t,\omega)$ as

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t,\omega)\, dt$$

We would like to know under what condition this quantity converges to the expected value of $X(t)$.

Example: $X(t) = A \quad \forall t$ where A is a zero-me random variable:

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^{T} A\, dt = A.$$

However, $E[X(t)] = m_X(t) = E[A] = 0$

So, the time average does not converge to $m_X(t)=0$. Note that the process is stationary but not ergodic.

- <u>Ergodic Theorem</u> for the time average of <u>WSS processes</u>.

Since the estimate $\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\omega)\,dt$ for $E[X(t)] = m_X(t)$ gives only one value, it is natural that mean ergodicity be of importance ~~only~~ for the case where $m_X(t)=m$.

$$E[\langle X(t)\rangle_T] = E\left[\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right] = \frac{1}{2T}\int_{-T}^{T} E[X(t)]\,dt =$$

So $\langle X(t)\rangle_T$ is an <u>unbiased</u> estimate of m

$$Var[\langle X(t)\rangle_T] = E[(\langle X(t)\rangle_T - m)^2]$$

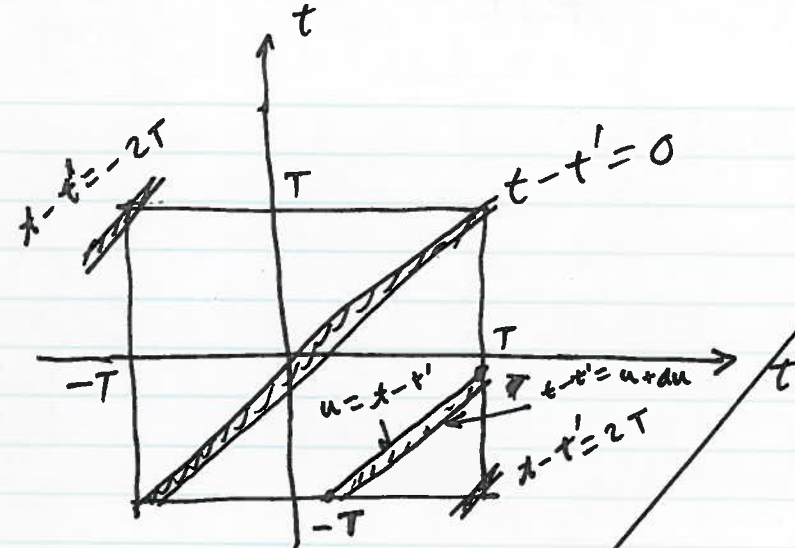
$$= E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t)-m)\,dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t')-m\right.\right.$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[X(t)-m][X(t')-m]\,dt\,dt$$

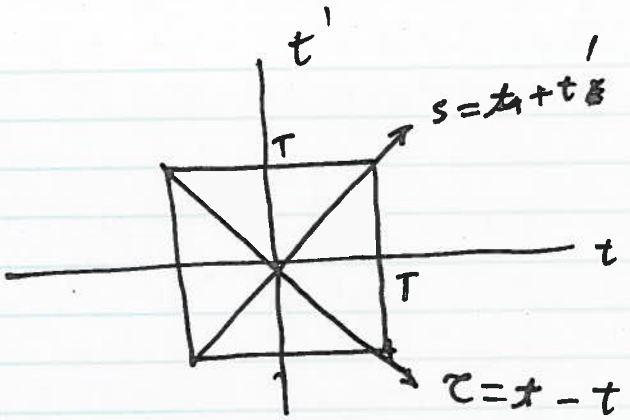
$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t,t')$$

when $X(t)$ is WSS

$$Var[\langle X(t)\rangle_T] = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t-t')\,dt\,dt'$$



~~Note that the integrand, $C_x(t-t')$ is a function of $t-t'$. Letting $u = t-t'$, we see that the integrand is constant on the line defined by $u = t-t'$;~~



Noting that the integrand is a function of $\tau = t - t'$, we make the change of variable

$$\begin{aligned} s &= t + t' \\ \tau &= t - t' \end{aligned} \Rightarrow |J| = \begin{vmatrix} \frac{\partial s}{\partial t} & \frac{\partial s}{\partial t'} \\ \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial t'} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

So

$$dt\,dt' = |J|^{-1} ds\,d\tau = \frac{1}{2} ds\,d\tau$$

So:

$$Var[\langle X(t) \rangle] = \frac{1}{4T^2} \int_{-2T}^{2T} \left[\int_{-(2T-|\tau|)}^{(2T-|\tau|)} \frac{1}{2} C_X(\tau)\, ds \right] d\tau$$

$$= \frac{1}{4T^2} \int_{-2T}^{2T} (2T - |\tau|)\, C_X(\tau)\, d\tau$$

$$= \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_X(\tau)\, d\tau$$

<u>Theorem</u>: For a WSS process

$$\lim_{T \to \infty} \langle X(t) \rangle = m$$

in the mean square sense if:

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_X(\tau)\,d\tau = 0$$

Then the process is said to be <u>ergodic</u> <u>in the mean</u> (or <u>mean</u> <u>ergodic</u>)

<u>Example</u>: A <u>random</u> <u>telegraph</u> signal

$$C_X(\tau) = e^{-2\alpha|\tau|}$$

$$Var[\langle X(t)\rangle_T] = \frac{2}{2T}\int_0^{2T}\left(1-\frac{u}{2T}\right)e^{-2\alpha u}\,du$$

$$< \frac{1}{T}\int_0^{2T} e^{-2\alpha\tau}\,d\tau = \frac{1-e^{-4\alpha T}}{2\alpha T}$$

as $T\to\infty$ the bound approaches <u>zero</u>.

So, ~~the~~ process is ergodic.

~~Example~~

Power Spectral Density

For WSS processes:

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

is called the PSD.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cos(2\pi f\tau)\, d\tau - j \int_{-\infty}^{\infty} R_X(\tau) \sin(2\pi f\tau)\, d$$

for real-valued random processes $R_X(\tau)$ is even so $R_X(\tau)\sin(2\pi f\tau)$ is odd. Therefore, the second integral is zero. So,

$$S_X(f) = \int_{-\infty}^{\infty} S_X(f) \cos(2\pi f\tau)\, d\tau$$

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df$$

Cross-power Spectral Density:

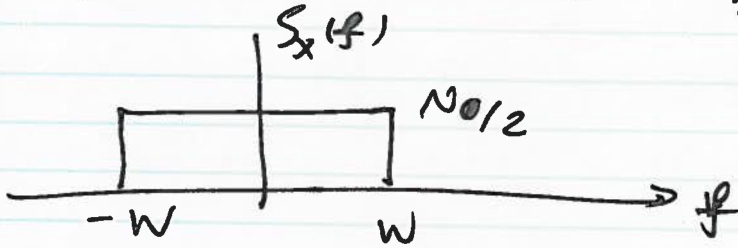
$$S_{X,Y}(\tau) = \mathcal{F}[R_{X,Y}(\tau)]$$

where

$$R_{X,Y}(\tau) = E[X(t+\tau)Y(t)]$$

Example:

A WSS process with $S_x(f) = \frac{N_0}{2} \quad |f| < W$

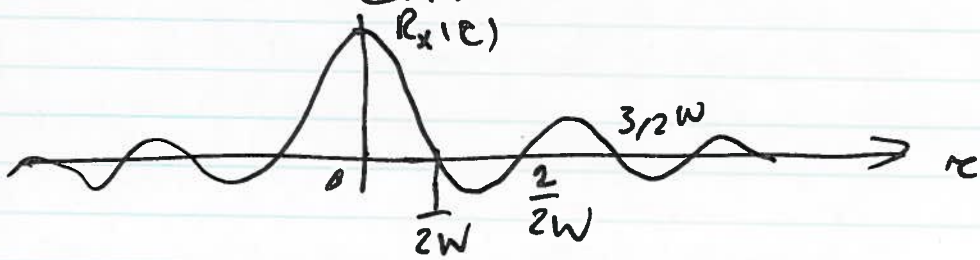


$$E[x^2(t)] = \int_{-W}^{W} \frac{N_0}{2}\, df = N_0 W$$

$$R_x(\tau) = \frac{1}{2} N_0 \int_{-W}^{W} e^{j2\pi f\tau}\, df$$

$$= \frac{N_0}{2} \frac{e^{-j2\pi Wt} - e^{j2\pi Wt}}{-j2\pi\tau}$$

$$= \frac{N_0 \sin(2\pi W\tau)}{2\pi W\tau}$$



$X(t)$ and $X(t+\tau)$ are uncorrelated if $\tau = \pm \frac{k}{2W}$

White noise

$$S_x(f) = \frac{N_0}{2} \quad \text{all } f$$

then $R_x(\tau) = \frac{N_0}{2}\delta(\tau)$

Example: PSD of random Telegraph signal

$$R_X(\tau) = e^{-2\alpha|\tau|}$$

$$S_X(f) = \int_{-\infty}^{0} e^{2\alpha\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-2\alpha\tau} e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{2\alpha - j2\pi f} + \frac{1}{2\alpha + j2\pi f} = \boxed{\frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}}$$

Discrete-Time random processes

$$S_X(f) = \mathcal{F}[R_X(k)] = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi f k}$$

$$R_X(k) = \int_{-1/2}^{1/2} S_X(f) e^{j2\pi f k} df$$

We only need to consider $-\frac{1}{2} < f < \frac{1}{2}$

since $S_X(f)$ is periodic, in f, with period 1.

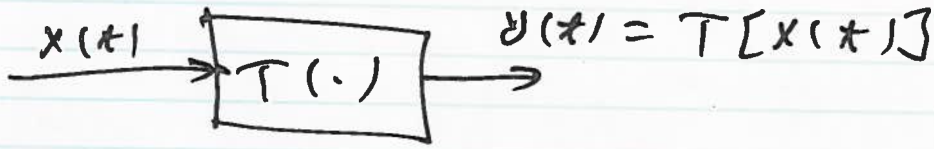
Example: X_n is a sequence of zero-mean uncorrelated r.v.'s with variance σ_X^2.

Find $S_X(f)$

$$R_X(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

So $S_X(f) = \sigma_X^2 \quad -\frac{1}{2} < f < \frac{1}{2}$ (it has equal pow ~~for~~ in each frequency

Response of Linear Systems to Random Signals



A system is linear if:

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x(t)] + \beta T[x_2(t)]$$

i.e., if superposition works.

A system is time invariant if

$$y(t) = T[x(t)] \Rightarrow y(t-\tau) = T[x(t-\tau)]$$

i.e., if the response to a signal delayed by τ is ~~the~~ a delayed (by τ) version of the response to the original signal.

Response of LTI filters

$$x(t) = \int_{-\infty}^{\infty} x(t-s)\,\delta(s)\,ds = \int_{-\infty}^{\infty} x(s)\,\delta(t-s)\,ds$$

i.e., any function can be written as a weighted sum of $\delta(\cdot)$ functions.

~~So~~ That is why, the input of an LTI to a delta function plays and important role.

$$h(t) = T[\delta(t)]$$

is called the impulse response of the system $T(\cdot)$.

The response of $T(\cdot)$ to $x(t)$ is

$$y(t) = T[x(t)] = T\left[\int_{-\infty}^{\infty} x(s)\delta(t-s)\,ds\right]$$

$$= \int_{-\infty}^{\infty} x(s)\,T[\delta(t-s)]\,ds$$

$$= \boxed{\int_{-\infty}^{\infty} x(s)\,h(t-s)\,ds} = \int_{-\infty}^{\infty} h(s)\,x(t-s)\,ds$$

or

$$y(t) = x(t) * h(t)$$

Let's now consider the output of an LTI to a random signal $X(t)$

$$X(t) \longrightarrow \boxed{h(\cdot)} \longrightarrow Y(t)$$

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] \cancel{= E\left[\int_{-\infty}^{\infty} h(s)X(t-s)\,ds\right.}$$

$$= E\left[\int_{-\infty}^{\infty} h(s)X(t-s)\,ds \int_{-\infty}^{\infty} h(r)X(t+\tau-r)\,dr\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\,E[X(t-s)X(t+\tau-r)]\,ds\,dr$$

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_x(\tau+s-r)\,ds\,dr$$

and

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau)e^{-j2\pi f\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_x(\tau+s-r)e^{-j2\pi f\tau}\,ds\,dr\,d\tau$$

let $u = \tau + s - r$

$$S_y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)R_x(u)e^{-j2\pi f(u-s+r)}\,ds\,dr\,du$$

$$= \int_{-\infty}^{\infty} h(s)e^{+j2\pi fs}\,ds\int_{-\infty}^{\infty} h(r)e^{-j2\pi fr}\,dr\int_{-\infty}^{\infty} R_x(u)e^{-j2\pi fu}\,du$$

$$= H^*(f)H(f)\,S_x(f) = \boxed{|H(f)|^2 S_x(f)}$$

Similarly,

$$R_{y,x}(\tau) = E[Y(t+\tau)X(t)] = E[X(t)\int_{-\infty}^{\infty} x(t+\tau-s)h(s)\,ds]$$

$$= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-s)]\,h(s)\,ds$$

$$= \int_{-\infty}^{\infty} R_x(\tau-s)h(s)\,ds = \boxed{R_x(\tau) * h(\tau)}$$

and

$$S_{Y,X}(f) = H(f)\,S_X(f)$$

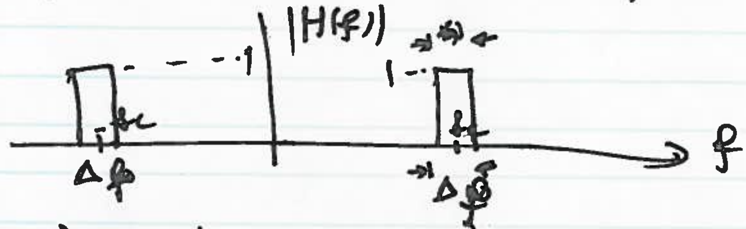
also

$$S_{X,Y}(f) = S^*_{Y,X}(f) = H^*(f)S_X(f)$$

since $R_{xy}(\tau) = R_{yx}(-\tau)$

Example

Find the power of the output of the filter

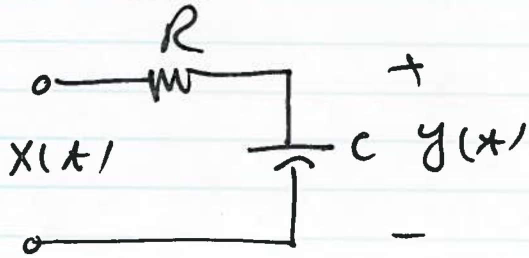


When the input $X(t)$ is applied to it

$$S_y(f) = |H(f)|^2 S_x(f) = S_x(f) \qquad f_c - \frac{\Delta f}{2} < |f| < f_c + \frac{\Delta f}{2}$$

$$E[y^2(t)] = \int_{-\infty}^{\infty} S_y(f)\, df = 2S_x(f_c)\,\Delta f$$

Example: A white Gaussian signal is applied to an RC circuit



Find the average power of the output

$$X(t) = RC\frac{dy(t)}{dt} + y(t)$$

$$X(f) = j2\pi fRC\,Y(f) + Y(f)$$

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$S_Y(f) = \frac{N_0/2}{1 + 4\pi^2 f^2 R^2 C^2}$$

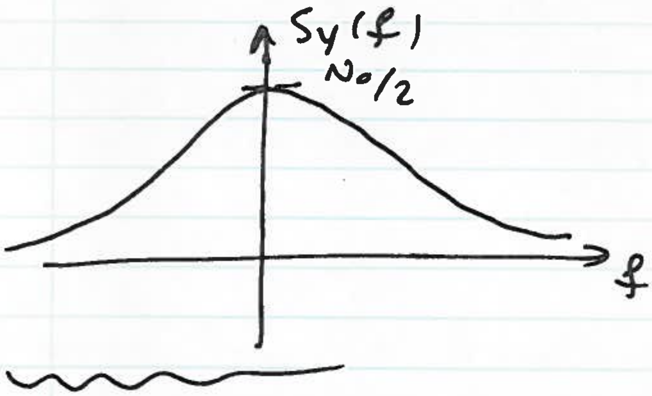
$$E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} \frac{N_0/2}{1+4\pi^2 f^2 R^2 C^2}\, df$$

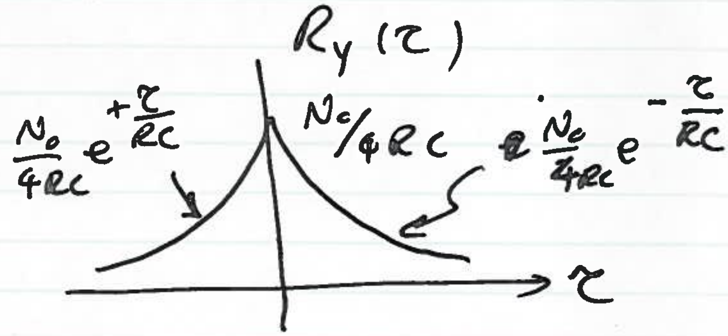
$$= N_0 \int_0^{\infty} \frac{1}{1+(2\pi RCf)^2}\, df$$

$$= \frac{N_0}{2\pi RC} \tan^{-1}(2\pi RCf)\Big|_0^{\infty} = \boxed{\frac{N_0}{4RC}}$$

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\frac{N_0/2}{1+4\pi^2 f^2 R^2 C^2}\right]$$

$$= \frac{N_0}{4RC} e^{-\frac{|\tau|}{RC}}$$





Markov Processes

A random process $X(t)$ is a Markov Process if the future of it given present is independent of the past, i.e.

$$P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k, \dots \quad X(t_1) = x_1]$$

$$= P[X(t_{k+1}) = x_{k+1} \mid X(t_k) = x_k]$$

For a discrete-valued process:

$$P[a < X(t_{k+1}) < b \mid X(t_k) = x_k, \dots \quad X(t_1) = x_1]$$

$$= P[a < X(t_{k+1}) < b \mid X(t_k) = x_k]$$

for continuous valued process:

$$f_{X(t_{k+1})}(x_{k+1} \mid X(t_k) = x_k, \dots X(t_1) = x_1] = f_{X(t_{k+1})}(x_{k+1} \mid X(t_k) = x_k)$$

Example

$$S_n = X_1 + X_2 + \cdots + X_n = S_{n-1} + X_n$$

~~$P[S_{n+1}$~~

$$P[S_{n+1} = s_{n+1} \mid S_n = s_n, \dots, S_1 = s_1] = P[X_{n+1} = s_{n+1} - s_n]$$

$$= P[S_{n+1} = s_{n+1} \mid S_n = s_n]$$

An integer valued Markov Process is called a <u>Markov Chain</u>.

For Markov chain the joint pmf is:

$$P[X(t_{k+1})=x_{k+1},\ X(t_k)=x_k,\ \cdots X(t_1)=x_1]$$
$$= P[X(t_{k+1})=x_{k+1} \mid X(t_k)=x_k]\, P[X(t_k)=x_k \mid X(t_{k-1})=x_{k-1}] \cdots$$
$$\cdots \times P[X(t_2)=x_2 \mid X(t_1)=x_1]\, P(X(t_1)=x_1]$$

Discrete-time Markov Chains

Let X_n for $n=0,1,\ldots$ form a Markov chain with $P_j(0) \triangleq P[X_0=j]$ $\quad j=0,1,2,\cdots$

The joint pmf of $X_0, X_1, \cdots X_n$ is

$$P[X_n=i_n, \cdots X_0=i_0] = P[X_n=i_n \mid X_{n-1}=i_{n-1}] \cdots P[X_1=i_1 \mid X_0=i]\, P[x$$

So, the pmf can be written as a product of <u>one step</u> <u>transition</u> probabilities of the form

$$P[X_{n+1}=j \mid X_n=i] = P_{ij} \qquad \text{all } n$$

(assuming that transition probabilities are independent of time, i.e., we have <u>homogeneous</u> <u>transition</u> <u>probabilities</u>.

Then the joint pmf is given as

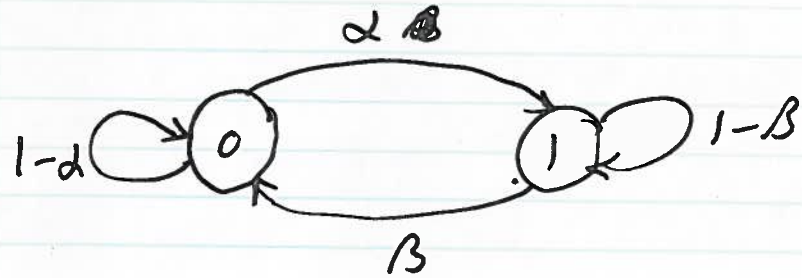
$$P[X_n = i_n, \cdots X_0 = i_0] = p_{i_0}(0)\, p_{i_0,i_1} \cdots p_{i_{n-1},i_n}$$

The process X_n is completely specified by the initial pmf $p_i(0)$ and the transition matrix

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & & & \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

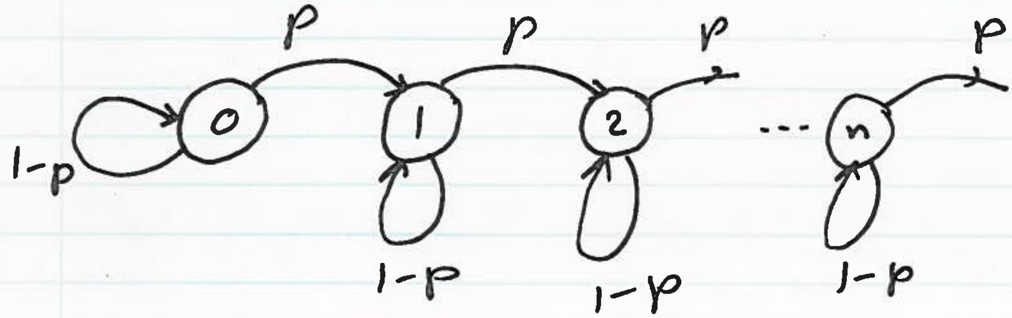
Where for each row $\sum_j p_{ij} = 1$

<u>Example</u> On-off source



$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Binomial Counting process



$$P = \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 0 & 1-p & p & 0 & 0 \cdots \\ 0 & 0 & 1-p & p & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The n-step Transition probabilities

Example of On-off source and $n=2$

$$P(X_{n+1}=0 \mid X_{n-1}=0) = P_{00}(2) = [(1-\alpha)^2 + \alpha\beta] = P(X_2=0 \mid X_0=0)$$

$$P_{01}(2) = [\alpha(1-\beta) + \alpha(1-\alpha)]$$

$$P_{10}(2) = [\beta(1-\beta) + \beta(1-\alpha)]$$

$$P_{11}(2) = [(1-\beta)^2 + \alpha\beta]$$

$$P(2) = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} = P^2$$

in general

$$\boxed{P(n) = P^n}$$

The state probabilities:

Let $\underline{P}(n) = \{p_j(n)\}$ be the state probabilities at time n. Then

$$p_j(n) = \sum_i P[X_n = j | X_{n-1} = i]\, P[X_{n-1} = i]$$

$$= \sum_i P_{ij}\, p_i(n-1)$$

i.e.,

$$\underline{P}(n) = \underline{P}(n-1)\underline{P}$$

By recursion

$$\underline{P}(n) = \underline{P}(n-1)P = P(n-2)P^2 = \cdots = \boxed{P(0)P^n}$$

At the steady state $\underline{P}(n) = P(0) = \underline{\pi}$ so:

$$\pi P^n = \pi \quad \text{for all } n$$

gives the initial probabilities.

~~Example: on-off~~

$$p_j(n) = \sum_i P_{ij}\, P_i(n-1)$$

as $n \to \infty \Rightarrow p_j(n) \to \pi_j$ and $P_i(n-1) \to p_i$. So,

$$\pi_j = \sum_i P_{ij}\, \pi_i \;\Rightarrow\; \underline{\pi} = \underline{\pi} P \quad \text{s.t.} \quad \sum_i \pi_i = 1$$

Example: Steady state probabilities of the On-off Source

$$\pi P = \pi \Rightarrow [\pi_0 \ \ \pi_1] \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} = [\pi_0 \ \ \pi_1]$$

$$\pi_0 = \pi_0(1-\alpha) + \pi_1 \beta$$

$$\pi_1 = \pi_0 \alpha + \pi_1 (1-\beta)$$

$$\cancel{\pi_0} = \cancel{\pi_0} - \alpha \pi_0 + \pi_1 \beta$$

$$\alpha \pi_0 = \pi_1 \beta = (1-\pi_0)\beta$$

$$\frac{\pi_0}{1-\pi_0} = \frac{\beta}{\alpha} \Rightarrow \boxed{\pi_0 = \frac{\beta}{\alpha+\beta} \text{ and } \pi_1 = \frac{\alpha}{\alpha+\beta}}$$

Continuous-time Markov Chains

Transition rate

Consider the time interval δ and define the rate at which process $X(t)$ jumps from state i to j as $\lim_{\delta \to 0} \frac{P_{ij}(\delta)}{\delta} = \gamma_{ij}$

Then, it can be shown that:

$$p_j'(t) = \sum_i \gamma_{ij} \, p_i(t)$$

This is called Chapman-Kolmogrov Equation.

Global Balance Equations

In the steady-state $p_j(t) \to p_j$.

So $p_j'(t) \to 0$ and, therefore,

$$\sum_i \gamma_{ij} p_i = 0 \quad \text{all } j.$$

These are called the global balance equations.

Queues

A queueing system consists of:

- An arrival process
- A service regime: No. of servers, their departure rule.
- A queue of certain size

A queue is defined by

$$a/b/m/K$$

where a is the arrival process, e.g.,

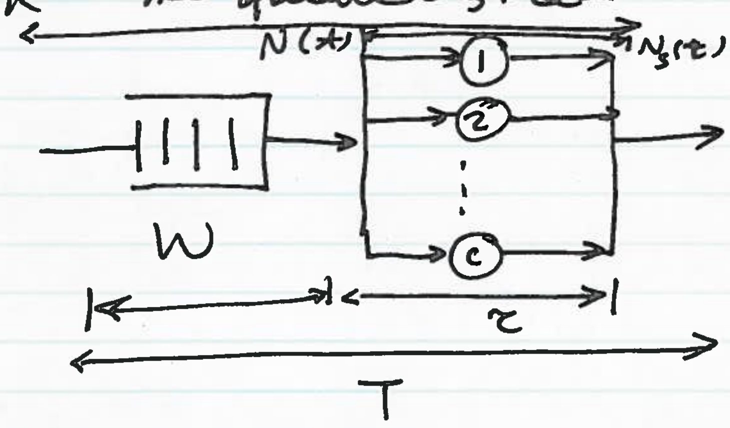
M for Poisson

b: The service time distribution, e.g.,

M exponential
D deterministic
G general

m denotes the number of servers

K the ~~queue size~~ total number of customers in the System.



$T = W + \tau$

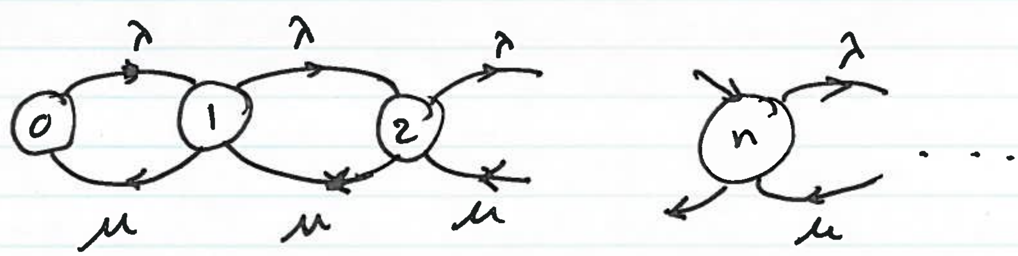
T is the total time in the system

W is the waiting time

τ is the service Time

Example

M/M/1/∞ or M/M/1 Queue



In a small period δ, Prob. of one Arrival is

1) $P[A(\delta)=1] = \frac{\lambda\delta}{1!} e^{-\lambda\delta} = \lambda\delta\left[1 - \frac{\lambda\delta}{1!} + \frac{(\lambda\delta)^2}{2!} - \cdots\right]$

$= \lambda\delta + o(\delta) \simeq \lambda\delta$

2)

$$P[A(\delta) \geq 2] = o(\delta)$$

3) Probability that there is one departure is:

$$P[\tau < \delta] = 1 - e^{-\mu\delta} = \mu\delta + o(\delta)$$

4) Probability that there is one arrival and one departure is

$$P[A(\delta) = 1, \tau \leq \delta] = P[A(\delta) = 1] P[\tau \leq \delta] = o(\delta)$$

Global Balance Equations are

$$\lambda P_0 = \mu P_1$$

(A) $$(\lambda + \mu) P_j = \lambda P_{j-1} + \mu P_{j+1} \quad j = 1, 2, \ldots$$

Note that (A) implies:

$$\lambda P_j - \mu P_{j+1} = \lambda P_{j-1} - \mu P_j \quad j = 1, 2, \ldots$$

that is $\lambda P_{j-1} - \mu P_j = \text{Constant} \quad j = 1, 2, \ldots$

But for $j = 1 \Rightarrow \lambda P_{j-1} - \mu P_j = 0$

So

$$\lambda P_{j-1} = \mu P_j \quad \text{all } j$$

$$P_j = \frac{\lambda}{\mu} P_{j-1}$$

Let $\rho = \frac{\lambda}{\mu}$ then

$$p_j = \rho\, p_{j-1} = \rho^2 p_{j-2} = \rho^3 p_{j-3} = \cdots = \rho^j p_0$$

$$\sum_j p_j = 1 \Rightarrow \sum_j \rho^j p_0 = 1$$

$$\Rightarrow p_0 \left[1 + \rho + \rho^2 + \cdots\right] = 1$$

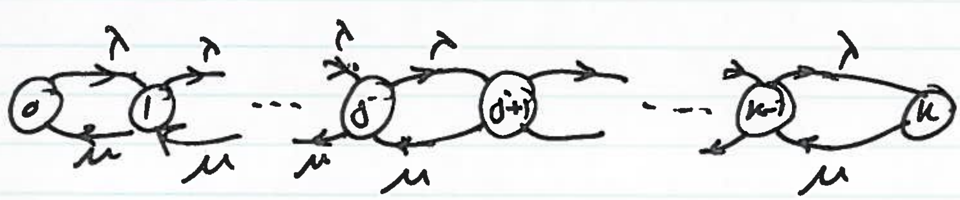
$$\Rightarrow p_0 \frac{1}{1-\rho} = 1 \Rightarrow p_0 = 1 - \rho$$

or

$$\boxed{p_j = (1-\rho)\rho^j} \qquad j = 0, 1, 2, \ldots$$

we need $\rho = \frac{\lambda}{\mu} < 1$ for stability.

M/M/1/K queue



$$\lambda p_0 = \mu p_1$$

$$(\lambda + \mu) p_j = \lambda p_{j-1} + \mu p_{j+1} \qquad j = 1, 2, \ldots, K-1$$

$$\mu p_K = \lambda p_{K-1}$$

Let $\rho = \frac{\lambda}{\mu}$

It is easy to show that in the steady state

$$P[N=j] = \frac{(1-\rho)\rho^j}{1-\rho^{K+1}} \quad j = 0, 1, 2, \dots, K$$

Blocking Probability, i.e., probability of rejecting a customer is:

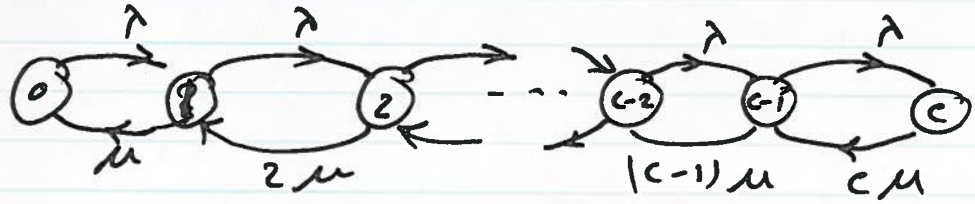
$$P[N=K] = \frac{(1-\rho)\rho^K}{1-\rho^{K+1}} = (1-\rho)\rho^K\left[1+\rho^{K+1} + (\rho^{K+1})^2 + \cdots\right.$$

if we had used an infinite queue system to approximate this probability, we had

$$P[N=K] = (1-\rho)\rho^K \quad \text{(of M/M/1 queue)}$$

which is the first term in the expansion.

M/M/c/c queue



$$p_0 \lambda = p_1 \mu$$

$$\lambda p_j + j\mu p_j = \lambda p_{j-1} + (j+1)\mu p_{j+1}$$

$$\lambda p_{j-1} - j\mu p_j = \text{Constant} = 0$$

$$p_j = \frac{1}{j}\left(\frac{\lambda}{\mu}\right) p_{j-1}$$

Let $a = \frac{\lambda}{\mu}$

Then

$$P_j = \frac{a^j}{j!} P_0$$

$$\sum_{j=0}^{c} P_j = P_0 \sum_{j=0}^{c} \frac{a^j}{j!} = 1 \Rightarrow P_0 = \frac{1}{\sum_{j=0}^{c} \frac{a^j}{j!}}$$

$$P_B = P[N=c] = \frac{a^c/c!}{1 + a + \frac{a^2}{2!} + \cdots + \frac{a^c}{c!}}$$

<u>Example</u>: A company has four telephone lines with an arrival rate of one call every two minutes. The duration of a call is 4 minutes on the average. What is the blocking probability?

$$\lambda = \frac{1}{2} \quad \text{and} \quad \mu = \frac{1}{4} \quad \Rightarrow a = \frac{\lambda}{\mu} = 2$$

$$P_B = \frac{2^4/4!}{1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}} = 0.095 \text{ or } 9.5\%$$

Little's Formula

$$E[N] = \lambda E[T]$$

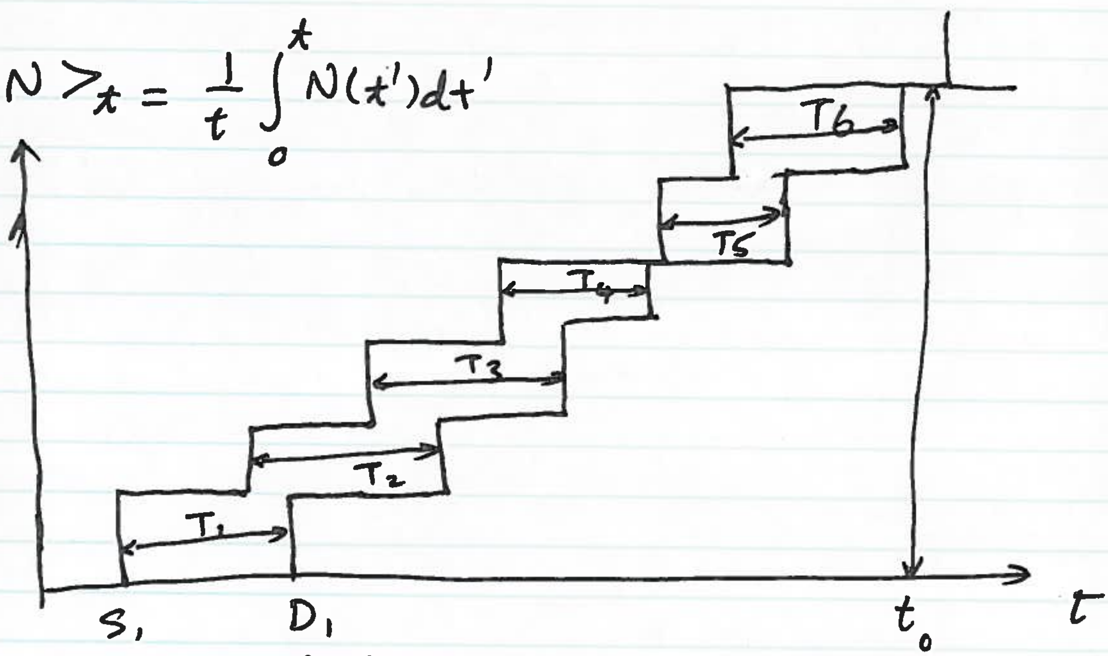
That is, the average number of customers in the system is the rate of arrival times the average time in the system.

Let $A(t)$ be the number of arrivals until time t.

$D(t) \triangleq$ number of departures until t.

Then
$$N(t) = A(t) - D(t)$$

$$\langle N \rangle_t = \frac{1}{t}\int_0^t N(t')dt'$$



$$\langle N(t) \rangle = \frac{1}{t}\sum_{i=1}^{A(t)} T_6$$

$$\langle \lambda \rangle_t = \frac{A(t)}{t} \Rightarrow \frac{1}{t} = \frac{\langle \lambda \rangle_t}{A(t)}$$

$$\langle N(t)\rangle_t = \langle \lambda \rangle_t \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i$$

But

$$\frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i = \langle T \rangle_t$$

So,

$$\langle N(t)\rangle_t = \langle \lambda \rangle_t \langle T \rangle_t$$

as $t \to \infty$, we assume that

$$\langle N(t)\rangle \to E[N]$$

$$\langle \lambda \rangle_t \longrightarrow \lambda$$

$$\langle T \rangle_t \to E[T]$$

So:

$$\boxed{E[N] = \lambda E[T]}$$

<u>Example</u>: M/M/1 queue

$$P_j = (1-\rho)\rho^j$$

$$E[N] = \sum_{j=0}^{\infty} j P_j = \sum_{j=0}^{\infty} (1-\rho) j \rho^j = \frac{\rho}{1-\rho}$$

$$E[T] = \frac{E[N]}{\lambda} = \frac{\rho/\lambda}{1-\rho} = \frac{1/\mu}{1-\rho} = \frac{E[\tau]}{1-\rho} = \frac{1}{\mu-\lambda}$$

Expected waiting time in the queue is:

$$E[W] = E[T] - E[\tau] = \frac{E[\tau]}{1-\rho} - E[\tau]$$

$$E[W] = \frac{\rho}{1-\rho} E[\tau] = \frac{\lambda/\mu}{\mu - \lambda}$$

Expected number in queue

$$E[N_q] = \lambda E[W] = \frac{\rho^2}{1-\rho}$$

Server utilization is

$$1 - P_0 = 1 - (1-\rho) = \boxed{\rho = \frac{\lambda}{\mu}}$$