This is called a Geometric R.V.

X Lecture 3: Sept. 22, 2004

4) Poisson Random Variable

$$P[N=k] = \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1, 2, \ldots$$

This models, e.g., the number of phone calls in a certain time interval or the number of packets needed to be transmitted in certain interval. α is the average number of occurences of that event in that time interval.

The Poisson probabilities can be derived from Binomial Distribution if we assume $\alpha = np$ and keep α fixed while letting $n \to \infty$, i.e.

$$p_k = \binom{n}{k} p^k (1-p)^{n-k} \longrightarrow \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1, \ldots$$

Continuous R.V.'s

exponential r.V.

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

Gaussian (Normal) r.V.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

CDF of Gaussian r.V. is:

$$F_X(x) = P[X \leq x] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(x'-m)^2}{2\sigma^2}} dx'$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt$$

$$= \Phi\left(\frac{x-m}{\sigma}\right)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

We usually use $Q(x)$ given as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt = 1 - \Phi(X)$$

Example :

Assume that in a communication system a ϕ is represented by voltage $-A$ and 1 is represented by a voltage level $+A$. Assume that the received voltage is

$Y = X + V$ where $V \sim \mathcal{N}(0,4]$.

a) Find the probability that zero is transmitted and the received voltage exceeds ϕ.

$$Y = -A + V$$

$$P[Y>0] = P[-A+V>0] = P[V>A]$$

$$= \frac{1}{\sqrt{2\pi}\,2}\int_{A}^{\infty} e^{-\frac{v^2}{2\times 4}}\,dV = \frac{1}{\sqrt{2\pi}}\int_{A/2}^{\infty} e^{-\frac{u^2}{2}}\,du = Q\left(\frac{A}{2}\right) \simeq$$

$$\frac{v}{2} = u \Rightarrow$$

b) Find A such that $P[Y>0] \leq 10^{-5}$

$$Q\left(\frac{A}{2}\right) \leq 10^{-5} \Rightarrow \frac{A}{2} \cong 4.3 \Rightarrow \boxed{A = 8.6}$$

Functions of a random variable

$Y = g(X)$ when X is a r.v. is itself another random variable. The CDF and pdf of $Y = g(X)$ can be found starting from

$$F_Y(y) = P[Y \leq y] = P[g(x) \leq y]$$

when $x = g^{-1}(y)$ exists and is unique, the problem is quite simple. When $g(x)$ is not one-to-one, the problem is usually more difficult.

<u>Example</u>: Find the pdf of $y = ax + b$ in terms of pdf of X.

$$F_Y(y) = P[Y \leq y] = P[ax+b \leq y]$$

$$= P\left[X \leq \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right) \qquad a > 0$$

or

$$= P\left[X \geq \frac{x-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right) \qquad x < 0$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{du} F_X(u) \times \frac{du}{dy} \qquad u = \frac{y-b}{a}$$

$$= f_X\left(\frac{y-b}{a}\right)\frac{1}{a} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0$$

or

$$-f_X\left(\frac{y-b}{a}\right)\frac{1}{a} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a < 0$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Example let X be $N(m, \sigma^2)$, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

then
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,|\sigma a|} e^{-\frac{(y-b-am)^2}{2(\sigma a)^2}}$$

<u>Example</u> $Y = X^2$

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}]$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \end{cases}$$

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \qquad y > 0$$

For $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \rightarrow$ ($X \sim N(0,1)$)

we have
$$f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \qquad y \geq 0$$

This is a Chi-square r.v. with one degree of freedom.

In general assume that $y = g(x)$ has n solutions $x_1, x_2, \ldots, x_n$, then

$$f_Y(y) = \sum_k \left. \frac{f_X(x)}{\left|dy/dx\right|} \right|_{x = x_k} = \boxed{\sum_k f_X(x) \left| \frac{dx}{dy} \right|}$$

Now, let's consider again $y = x^2$.

We have $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$

So:

$$f_X(\sqrt{y}) \frac{d}{dy}(\sqrt{y}) + f_X(-\sqrt{y}) \frac{d}{dy}(-\sqrt{y})$$

$$f_Y(y) = \cancel{\frac{f_X(\sqrt{y})}{\frac{d}{dy}(\sqrt{y})} + \frac{f_X(-\sqrt{y})}{\frac{d}{dy}(-\sqrt{y})}}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

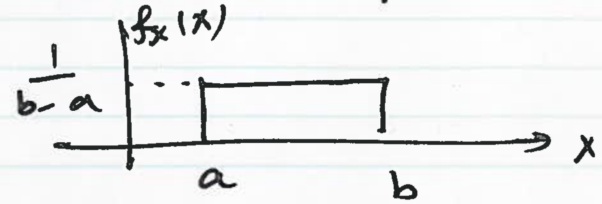
Expected value of a random variable:

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$ for continuous r.v.

and

$$E[X] = \sum_k x_k \, P_X(x_k)$$

Example : Uniform r.v.



$$E[X] = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\cdot\frac{1}{2}[b^2 - a^2] = \frac{b+a}{2}$$

When $f_X(x)$ is symmetric about $x = m$.

Then, we have

$$\int_{-\infty}^{\infty} (m - x) f_X(x)\,dx = 0$$

$$m = m\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} x f_X(x)\,dx = E[X]$$

So $\boxed{E[X] = m}$

The arrival time of packets to a queue (or in general the arrival time of customers at a service centre) has exponential pdf.

$$f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

$$E[X] = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx = \underbrace{-xe^{-\lambda x}\Big|_0^{\infty}}_{0} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda}$$

So, the mean inter-arrival time is $\frac{1}{\lambda}$ which makes sense since λ is the packet arrival rate.

For a function $Y = g(X)$ of r.v. X, we have,

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

Example: $y = a\cos(\omega t + \Phi)$ where Φ is uniformly distributed in $[0, 2\pi)$. Find the expected value of Y and y^2 (the power of Y)

$$E[Y] = \int_0^{2\pi} \frac{1}{2\pi} a \cos(\omega t + \phi)\, d\phi = -\frac{a}{2\pi} \sin(\omega t + \phi)\Big|_0^{2\pi}$$

$$= -\frac{a}{2\pi}[\sin(\omega t + 2\pi) - \sin(\omega t)] = 0$$

$$E[Y^2] = E[a^2 \cos^2(\omega t + \phi)] = E\left[\frac{a^2}{2} + \frac{a^2}{2}\cos(2\omega t + 2\phi)\right]$$

$$= \frac{a^2}{2}$$

Variance of a random variable.

$$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Variance shows the deviation of a r.v. about its mean, i.e.,

$$Var[X] = E[D^2] \text{ where}$$

$$D = X - E[X].$$

Variance is the mean-squared error when one approximates a r.v. by its mean.

Example: Uniform r.v.

$$Var(X) = \int_a^b \frac{1}{b-a} \left(x - \frac{a+b}{2}\right)^2 dx$$

let $y = x - \frac{a+b}{2}$

$$Var(x) = \frac{1}{b-a} \int_{-\frac{(b-a)}{2}}^{\frac{b-a}{2}} y^2 \, dy = \frac{(b-a)^2}{12}$$

For a Gaussian random variable, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = 1$$

or

$$\int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \sqrt{2\pi}\,\sigma$$

Take derivative, of both sides, w.r.t. σ:

$$\int_{-\infty}^{\infty} \frac{(x-m)^2}{\sigma^3} e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \sqrt{2\pi}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-m)^2 e^{-\frac{(x-m)^2}{2\sigma^2}}\, dx = \sigma^2$$

or

$$Var[X] = \sigma^2.$$

some: Properties of variance

$$Var[c] = 0$$

$$Var[X+c] = Var[X]$$

$$Var[cX] = c^2\, Var[X]$$