X Lecture 4, Sept. 29, 04

n-th moment of a random variable:

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

The Markov inequality

What do mean and variance say about $P[|X| \geq t]$?

For ~~positive~~ non-negative r.v. X, we have

$$P[X \geq a] \leq \frac{E[X]}{a} \quad \text{for } a > 0$$

Proof:

$$E[X] = \int_0^{\infty} x f_X(x)\,dx = \int_0^{a} x f_X(x)\,dx + \int_a^{\infty} x f_X(x)\,dx$$

$$\geq \int_a^{\infty} x f_X(x)\,dx \geq \int_a^{\infty} a f_X(x)\,dx = a P[X \geq a]$$

so,

$$\boxed{P[X \geq a] \leq \frac{E[X]}{a}}$$

Chebyshev inequality

$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2}$$

This is a consequence of Markov inequality

$$P[|X-m| \geq a] = P[(X-m)^2 \geq a^2] \leq \frac{E[(X-m)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

Characteristic Function

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

$\Phi_X(\omega)$ is the Fourier transform of $f_X(x)$

So,

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} d\omega$$

<u>example</u>: The characteristic function of an exponentially distributed r.v. is:

$$\Phi_X(\omega) = \int_0^{\infty} \lambda e^{-\lambda x} e^{j\omega x} dx = \int_0^{\infty} \lambda e^{-(\lambda - j\omega)x} dx.$$

$$= \frac{\lambda}{\lambda - j\omega}$$

For discrete r.v.'s we have

$$\Phi_X(\omega) = \sum_k p_X(x_k) e^{j\omega x_k}$$

if x_k are integer-valued, we have,

$$\Phi_X(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

and $$P_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_X(\omega) e^{-j\omega k} d\omega$$

example: for a geometric r.v., we have:

$$\Phi_X(\omega) = \sum_{k=0}^{\infty} p(1-p)^k e^{j\omega k} = p \sum [(1-p)e^{j\omega}]^k$$

$$= \frac{p}{1-(1-p)e^{j\omega}}$$

Relationship between the Characteristic Function and the moments of a r.v.,

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \left[1 + j\omega x + \frac{(j\omega x)^2}{2!} + \cdots\right] dx$$

$$= 1 + j\omega E[X] + \frac{(j\omega)^2}{2!} E[X^2] + \cdots + \frac{(j\omega)^n}{n!} E[X^n] + \cdots$$

So:

$$\Phi_X(\omega)\Big|_{\omega=0} = 1 \quad \left(\text{obvious since } \Phi_X(0) = \int_{-\infty}^{\infty} f_X(x)\,dx\right)$$

$$\frac{d}{d\omega}\Phi_X(\omega)\Big|_{\omega=0} = jE[X]$$

$$\vdots$$

$$\frac{d^n}{d\omega^n}\Phi_X(\omega)\Big|_{\omega=0} = (j)^n E[X^n]$$

So:

$$\boxed{E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega) \Big|_{\omega=0}}$$

Probability ~~Moment~~ generating function:

For non-negative random variables, it is more convenient to use z-transform or Laplace Transform:

$$G_N(z) = E[z^N] = \sum_{k=0}^{\infty} p_N(k) z^k$$

It is easy to ~~shoo~~ show that

$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \Big|_{z=0}$$

we can also generate moments as follows:

$$\frac{d}{dz} G_N(z)\Big|_{z=1} = \sum_{k=0}^{\infty} p_N(k) k z^{k-1} \Big|_{z=1} = \sum_{k=0}^{\infty} k\, p_N(k) = E[N]$$

$$\frac{d^2}{dz^2} G_N(z)\Big|_{z=1} = \sum_{k=0}^{\infty} p_N(k)\, k(k-1) z^{k-2} \Big|_{z=1}$$

$$= \sum_{k=0}^{\infty} k(k-1) p_N(k) = E[N(N-1)]$$

$$= E[N^2] - E[N].$$

~~For~~
example: For a Poisson random variable:

$$G_N(z) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} z^k = e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!}$$

$$= e^{-\alpha} \cdot e^{\alpha z} = \boxed{e^{\alpha(z-1)}}$$

$$G_N'(z) = \alpha e^{\alpha(z-1)}$$

and

$$G_N''(z) = \alpha^2 e^{\alpha(z-1)}.$$

So

$$E[N] = G_N'(1) = \boxed{\alpha}$$

~~$E[N^2] =$~~

$$Var[N] = G_N''(1) + G_N'(1) - (G_N'(1))^2$$

$$= \cancel{\alpha^2} + \alpha - \cancel{\alpha^2} = \boxed{\alpha}$$