× Lecture 7, Oct. 20, 2004

Expected value of functions of r.v.'s:

$z = g(x_1, \cdots x_n)$

$$E[z] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \cdots x_n) f_{X_1, \cdots X_n}(x_1, \cdots x_n)\, dx_1 \cdots dx_n$$

for the discrete case

$$E[z] = \underset{\substack{\text{all possible} \\ \text{values of} \\ \underline{x}}}{\sum \cdots \sum} g(x_1, \cdots x_n) P_{X_1, \cdots X_n}(x_1, \cdots x_n)$$

Example

$z = X_1 + X_2 + \cdots + X_n$

then

$$E[z] = E[X_1 + \cdots X_n] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (X_1 + \cdots X_n) f_{X_1, \cdots X_n}(x_1, \cdots x_n)\, d\underline{x}$$

$$= E[X_1] + \cdots + E[X_n]$$

Example:

$z = X_1 X_2 \cdots X_n$

$$E[z] = \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} X_1 X_2 \cdots X_n f_{\underline{X}}(\underline{x})\, dx_1 \cdots dx_n$$

only if $X_1 \cdots X_n$ are independent, we have

$$E[X_1 X_2 \cdots X_n] = E[X_1] \cdots E[X_n]$$

the (j,k)-th moment of two r.v.'s X and Y is given as

$$E[X^j Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{X,Y}(x,y)\,dx\,dy \quad \text{continuous}$$

or

$$= \sum\sum x^j y^k p_{X,Y}(x,y) \quad \text{discrete}$$

for $j = k = 1$, we call the moment the correlation

$$E[XY] = \iint xy\, f_{X,Y}(x,y)\,dx\,dy$$

if $E[XY] = 0 \Rightarrow$ we call the two r.v.'s ~~uncorrelated~~ orthogonal.

The central moment of X and Y is defined as

$$E[(X - E(X))^j (Y - E(Y))^k]$$

when $j = 2$ and $k = 0$ we get $Var(X)$.

for $j=k=1$, we get

$$\mathrm{Cov}(X,Y) = E[(X-E(X))(Y-E(Y))]$$

$$= E[XY] - E[X]E[Y].$$

The <u>correlation coefficient</u> of X and Y is defined as

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY]-E[X][Y]}{\sigma_X \sigma_Y}$$

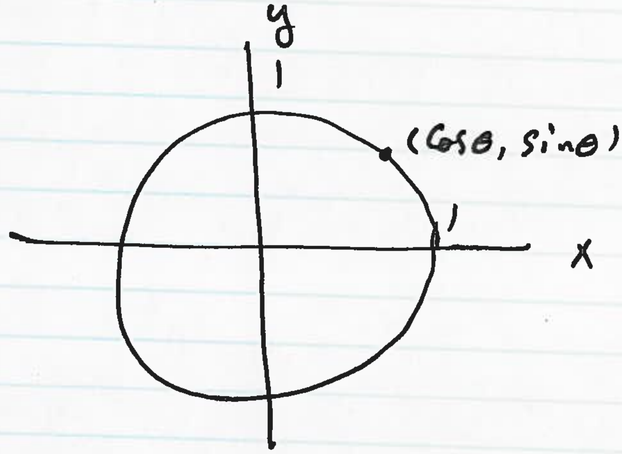
if $\rho_{X,Y} = 0$ we call the r.v.'s <u>uncorrelated</u>

if two random variables are independent then their $\mathrm{Cov}(X,Y) = 0 \Rightarrow$ uncorrelated.

The converse is not always true. It is only always true in the case of <u>Gaussian random variables</u>.

Example:

$X = \cos\theta$ and $Y = \sin\theta$ with θ uniform on $[0, 2\pi]$



It is clear that X and Y are not independent.

However:

$$E[XY] = E[\sin\theta \cos\theta] = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \cos\theta \, d\theta$$

$$= \frac{1}{4}\pi \int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

and

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y] = 0$$

So X and Y are correlated but not independent.

Joint Characteristic Function

$$\Phi_{X_1 \cdots X_n}(\omega_1, \cdots \omega_n) = E[e^{j(\omega_1 x_1 + \cdots + \omega_n x_n)}]$$

For 2 variables:

$$\Phi_{X,Y}(\omega_1, \omega_2) = E[e^{j(\omega_1 x + \omega_2 y)}]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, e^{j(\omega_1 x + \omega_2 y)}\, dx\, dy$$

equivalently,

$$f_{X,Y}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{X,Y}(\omega_1, \omega_2)\, e^{-j(\omega_1 x + \omega_2 y)}\, d\omega_1\, d\omega_2$$

We have

$$\Phi_X(\omega) = \Phi_{X,Y}(\omega, 0) \quad \text{and} \quad \Phi_Y(\omega) = \Phi_{X,Y}(0, \omega)$$

For two independent variables:

$$\Phi_{X,Y}(\omega_1, \omega_2) = E[e^{j\omega_1 X + j\omega_2 Y}] = E[e^{j\omega_1 X} \cdot e^{j\omega_2 Y}]$$

$$= E[e^{j\omega_1 X}]\, E[e^{j\omega_2 Y}] = \Phi_X(\omega_1)\, \Phi_Y(\omega_2)$$

Let $Z = aX + bY$

$$\Phi_Z(\omega) = E[e^{j\omega(aX + bY)}] = \Phi_{X,Y}(a\omega, b\omega)$$

Jointly Gaussian Random variables

Consider a vector of random variables:

$\underline{X} = (X_1, X_2, \cdots X_n)$ each with mean $m_i = E[X_i]$ $i=1,\ldots$

and the covariance matrix:

$$K = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,\cdots X_n) \\ \vdots & & & \vdots \\ Cov(X_n,X_1) & \cdots & \cdots & Var(X_n) \end{bmatrix}$$

Let

$$\underline{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$ be the ~~vect~~ mean vector

and ~~of~~ $\underline{x} = (x_1 \cdots x_n)^T$

Then, we have:

$$f_{\underline{X}}(\underline{x}) = f_{X_1 \cdots X_n}(x_1, \cdots x_n) = \frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-m)^T K^{-1} (x-m)\right]$$

where $|K|$ is the determinant of K and ~~K~~ $(\cdot)^T$ denotes transpose.

For $n=2$, denote the variables by X and Y then

$$K = \begin{bmatrix} \sigma_x^2 & \rho_{x,y}\sigma_x\sigma_y \\ \rho_{x,y}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

and

$$\underline{m} = \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

$$K^{-1} = \frac{1}{\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)} \begin{bmatrix} \sigma_y^2 & -\rho_{xy}\sigma_x\sigma_y \\ -\rho_{xy}\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}$$

$$|K| = \sigma_x^2\sigma_y^2(1-\rho_{x,y}^2)$$

$$\cancel{f_{x,y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}\exp\left[-\frac{1}{2}\,[x\ y]\,\frac{\begin{bmatrix} \sigma_y^2 & -\rho_{xy}\sigma_x\sigma_y \\ -\rho_{xy}\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}}{\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\begin{bmatrix} x \\ y \end{bmatrix}\right]}$$

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}\exp\left[-\frac{1}{2}\,\frac{1}{\sigma_x^2\sigma_y^2(1-\rho_{xy}^2)}\,[x-m_x,\ y-m_y]\,K^{-1}\begin{bmatrix} x-m_x \\ y-m_y \end{bmatrix}\right]$$

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{xy}^2}}\exp\left[-\frac{1}{2(1-\rho_{xy}^2)}\left[\left(\frac{x-m_x}{\sigma_x}\right)^2 - 2\rho_{xy}\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y-m_y}{\sigma_y}\right)^2\right]\right]$$

The marginal density of x is:

$$f_X(x) = \frac{e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\,\sigma_x}$$

and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\exp\left[\frac{-1}{2(1-\rho_{xy}^2)\sigma_x^2}\left[x - \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-m_y) - m_x\right]^2\right]}{\sqrt{2\pi\sigma_x^2(1-\rho_{xy}^2)}}$$

From this, we see that:

$$\boxed{\begin{aligned} E[X|Y] &= \rho_{xy}\frac{\sigma_x}{\sigma_y}(y-m_y) + m_x \\ Var(X|Y) &= \sigma_x^2(1-\rho_{xy}^2) \end{aligned}}$$

<u>Linear Transformation of</u> Gaussian Random variables

let $$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{n/2}|K|^{1/2}}\exp\left[-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})\right]$$

and let

$$\underline{Y} = A\,\underline{X}$$

then

$$f_{\underline{y}}(\underline{y}) = \frac{f_{\underline{x}}(A^{-1}\underline{y})}{|A|}$$

$$= \frac{1}{(2\pi)^{n/2}|A||K|^{1/2}} \exp\left[-\frac{1}{2}(A^{-1}\underline{y}-\underline{m})^T K^{-1}(A^{-1}\underline{y}-\underline{m})\right]$$

$$A^{-1}\underline{y}-\underline{m} = A^{-1}(\underline{y}-A\underline{m})$$

and

$$(A^{-1}\underline{y}-\underline{m})^T = (\underline{y}-A\underline{m})^T(A^{-1})^T$$

So:

$$(A^{-1}\underline{y}-\underline{m})^T K^{-1}(A^{-1}\underline{y}-\underline{m}) =$$

$$= (\underline{y}-A\underline{m})(A^{-1})^T K^{-1}(A^{-1}\underline{y}-A\underline{m})$$

$$= (\underline{y}-A\underline{m})(AKA^T)^{-1}(\underline{y}-A\underline{m})$$

let

$$\hat{\underline{m}} = A\underline{m}$$

and

$$\underline{C} = AKA^T$$

then

$$f_{\underline{y}}(\underline{y}) = \frac{1}{(2\pi)^{n/2}|C|^{1/2}} \exp\left[-\frac{1}{2}(\underline{y}-\hat{\underline{m}})^T C^{-1}(\underline{y}-\hat{\underline{m}})\right]$$

That is, ~~the~~ $\underline{y}$ is Gaussian with the mean vector $\hat{\underline{m}}$ and covariance matrix $\underline{C}$

Since $\underline{K}$ is symmetric (why?) ~~then~~ it is always possible to find a matrix $\underline{A}$ such that

$$\Lambda = AKA^T$$

is diagonal.

In such a case,

$$f_{\underline{y}}(\underline{y}) = \frac{1}{(2\pi)^{n/2}|\Lambda|} e^{-\frac{1}{2}(y-\hat{m})^T\Lambda^{-1}(y-\hat{m})}$$

$$= \frac{1}{\left[(2\pi\lambda_1)(2\pi\lambda_2)\cdots(2\pi\lambda_n)\right]^{1/2}} \exp\left[-\frac{1}{2}\sum\frac{(y_i-\hat{m}_i)^2}{\lambda_i}\right]$$

That is, we could transform $\underline{x}$ into n independent random variables with means $\hat{m}_i$ and variance λ_i. This ~~is~~ is called Karhunen-Loève Transform.