Reading Assignment:

Read Mean Square Estimation

and

Linear Prediction

X Lecture 8, Oct. 27, 04

Sum of Random Variables

Motivation: Counting, e.g., probability of error.

Let $X_1, X_2, \cdots X_n$ be random variables and

$$S_n = X_1 + X_2 + \cdots + X_n$$

then

$$E[S_n] = E[X_1] + \cdots + E[X_n]$$

$$\begin{aligned}
Var[S_n] &= Var(X_1 + X_2 + \cdots + X_n) \\
&= E\left[\sum_{i=1}^{n}(X_i - E[X_i]) \sum_{j=1}^{n}(X_j - E[X_j])\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[[(X_i - E(X_i)][X_j - E(X_j)]\right] \\
&= \sum_{i=1}^{n} Var(X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{n}\sum_{j=1}^{n} Cov(X_i, X_j)
\end{aligned}$$

For two variables X and Y Let $Z = X + Y$

$$Var(Z) = E[(Z - E(Z))^2] = E[(X + Y - E(X) - E(Y))^2]$$

$$= E[(X - \bar{X})^2] + E[(Y - \bar{Y})^2]$$

$$+ 2E[(X - \bar{X})(Y - \bar{Y})]$$

$$= Var(X) + Var(Y) + 2\, Cov(X, Y).$$

Example: Sum of n i.i.d. $N(\mu, \sigma^2)$ variables:

$$E[S_n] = E[X_1] + \cdots + E[X_n] = n\mu$$

$$Var[S_n] = n\, Var(X_i) = n\sigma^2$$

$$f_S(s) = \frac{1}{\sqrt{2\pi n}\,\sigma} e^{-\frac{(s - n\mu)^2}{2n\sigma^2}}$$

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pdf of sums of independent random variables:

Let $S_n = X_1 + X_2 + \cdots + X_n$

then

$$\Phi_{S_n}(\omega) = E[e^{j\omega S_n}] = E[e^{j\omega(X_1 + \cdots + X_n)}]$$

$$= \Phi_{X_1}(\omega)\Phi_{X_2}(\omega) \cdots \Phi_{X_n}(\omega)$$

and

$$f_{S_n}(s) = \mathcal{F}^{-1}\{\Phi_{X_1}(\omega) \cdots \Phi_{X_n}(\omega)\}$$

~~Example: The same as the previous one, i.e.,~~
~~n r.v.'s N(~~

$\cancel{\Phi_{X_i}}$

Example: $X_1, \cdots X_n$ where $X_i \sim N(m_i, \sigma_i^2)$

$$\Phi_{X_i}(\omega) = e^{j\omega m_i - \frac{\omega^2\sigma_i^2}{2}}$$

$$\Phi_{S_n}(\omega) = \prod_{i=1}^{n} e^{j\omega m_i - \frac{\omega^2\sigma_i^2}{2}}$$

$$= e^{j\omega(m_1 + m_2 + \cdots + m_n) - \omega^2(\sigma_1^2 + \cdots + \sigma_n^2)/2}$$

So, the sum of n independent Gaussian r.v.'s is Gaussian with the mean $m_1 + \cdots + m_n$ and variance $\sigma_1^2 + \cdots + \sigma_n^2$.

pdf of sum of m independent identically distributed random variables:

$$\Phi_{S_m}(\omega) = (\Phi_X(\omega))^m$$

and the pdf of S_n is $\mathcal{F}^{-1}[\Phi_{S_n}(\omega)]$

example: Find the pdf of the sum of n independent exponentially distributed random variables all with parameter λ.

$$\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega} = \mathcal{F}[\alpha e^{-\alpha x}] = \int_0^\infty \lambda e^{-\lambda x} e^{j\omega x}\,dx$$

$$\Phi_{S_n}(\omega) = \left\{\frac{\lambda}{\lambda - j\omega}\right\}^m$$

$$f_{S_n}(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{m-1}}{(m-1)!} \qquad x > 0$$

this is an m-erlang random variable.

When dealing with integer-valued random variables, we use the probability generating function:

$$G_N(z) = E[z^n] = \sum_n z^n p[N=n] = \sum_n z^n p_N(n)$$

$$p_N(n) = \frac{1}{n!}\frac{d^n}{dz^n} G_N(z)\Big|_{z=0}$$

also

$$\frac{d}{dz} G_n(z)\Big|_{z=\cancel{0}} = \sum_{n=0}^{\infty} p_N(n)\, n z^{n-1}\Big|_{z=1} = \sum_{n=0}^{\infty} n\, p_N(n) = E[N]$$

and

$$\frac{d^2}{dz^2} G_N(z)\Big|_{z=1} = E[N^2] - E[N]$$

or

$$Var(N) = E[N^2] - E^2[N] = G_N''(1) + G_N'(1) - (G_N'(1))^2$$

For $N = X_1 + X_2 + \cdots + X_n$ where X_i are ~~iid~~ independent discrete random variables.

$$G_N(z) = E[z^{X_1 + \cdots X_n}] = E[z^{X_1}]E[z^{X_2}] \cdots E[z^{X_n}]$$

$$= G_{X_1}(z)\, G_{X_2}(z) \cdots G_{X_n}(z)$$

example

Find the generating function of the sum of n independent geometrically distributed random variables: $p_X(n) = p(1-p)^{n-1}$

$$G_X(z) = E[z^X] = \sum_{n=0}^{\infty} z^n p_X(n) = \sum_{n=0}^{\infty} z^n p(1-p)^{n-1}$$

$$= pz \sum_{n=0}^{\infty} [z(1-p)]^n = \frac{pz}{1-(1-p)z}$$

So

$$G_N(z) = \left[\frac{pz}{1-(1-p)z}\right]^n$$

$$p_N(k) = \binom{k-1}{n-1} p^n (1-p)^{k-n} \quad \text{for } k = n, n+1, \ldots$$

a negative Binomial distribution.

example:

Find the pdf of the sum of n independent Bernouli random variables:

$$p_0 = q = 1 - p \qquad p_1 = p$$

$$G_X(z) = E[z^X] = \sum_{k=0}^{\infty} z^k p_X(k) = z^0 p_X(0) + z^1 p_X(1)$$

$$G_X(z) = (1-p) + pz = q + pz$$

$$G_N(z) = [G_X(z)]^n = [q + pz]^n$$

$$P_N(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n$$

Binomial random variable.

Let
$$M_n = \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$
be the sampled mean of the random variable X.

It has a mean:

$$E[M_n] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i\right] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} \sum_{i=1}^{n} E[X] = \mu$$

and variance:

$$E[(M_n - \mu)^2] = E\left[\left(\frac{S_n - E(S_n)}{n}\right)^2\right] = \frac{1}{n^2} E[(S_n - E(S_n))^2]$$

$$= \frac{1}{n^2} Var(S_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

From Chebyshev inequality, we have

$$P[|M_n - E[M_n]| \geq \varepsilon] \leq \frac{Var(M_n)}{\varepsilon^2}$$

or

$$P[|M_n - \mu| \geq \varepsilon] \leq \frac{\sigma^2}{n\varepsilon^2}$$

or

$$P[|M_n - \mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

So, with a probability tending to one (with the number of samples), the sample mean is as close as we wish to the actual mean.

This is the weak law of large numbers: i.e.,

$$\lim_{n \to \infty} P[|M_n - \mu| < \varepsilon] = 1$$

The strong law of large number says that with probability one any sample mean tends to mean.

$$P[\lim_{n \to \infty} M_n = \mu] = 1.$$

This is stronger in the sense that it assures that any sequence $M_n = \frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n$ tends to μ as $n \to \infty$.

The <u>central limit theorem</u>

Let $X_1, \cdots X_n$ be i.i.d. random variables.

$S_n = X_1 + \cdots + X_n$ has mean $n\mu$

and variance $n\sigma^2$. Let

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

then as $n \to \infty$ the distribution of Z_n tends to standard Gaussian, i.e.,

$$\lim_{n\to\infty} P[Z_n \leq z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx = 1 - Q(z)$$

Proof of the central limit theorem:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)$$

$$\Phi_{Z_n}(\omega) = E[e^{j\omega Z_n}] = E[\exp\{\frac{j\omega}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)]$$

$$= E[\prod_{i=1}^{n} e^{j\omega(X_i - \mu)/\sigma\sqrt{n}}]$$

$$= \prod_{i=1}^{n} E[e^{j\omega(X_i - \mu)/\sigma\sqrt{n}}] \quad \text{because of indepen}$$

$$= \{E[e^{j(X-\mu)/\sigma\sqrt{n}}]\}^n \quad \text{because of identical distribution.}$$

We have

$$E[e^{jw(X-\mu)/\sigma\sqrt{n}}] =$$

$$= E\left[1 + \frac{jw}{\sigma\sqrt{n}}(X-\mu) + \frac{(jw)^2}{2!\,n\sigma^2}(X-\mu)^2 + R(w)\right]$$

$$= 1 + \frac{jw}{\sigma\sqrt{n}}\underbrace{E[(X-\mu)]}_{0} + \frac{(jw)^2}{2!\,n\sigma^2}\underbrace{E[(X-\mu)^2]}_{\sigma^2} + E[R(w)]$$

or

$$E[e^{jw(X-\mu)/\sigma\sqrt{n}}] = 1 - \frac{w^2}{2n} + E[R(w)]$$

as $n \to \infty$, $E[R(w)]$ becomes negligible compared to $w^2/2n$.

$$\Phi_{Z_n}(w) = \left\{\left(1 - \frac{w^2}{2n}\right)\right\}^n$$

$$\lim_{n\to\infty} \Phi_{Z_n}(w) = e^{-w^2/2} \qquad \lim_{n\to\infty}\left(1-\frac{a}{n}\right)^n = e^{-a}$$

So Z_n is Gaussian with <u>mean zero</u> and <u>variance</u> one.

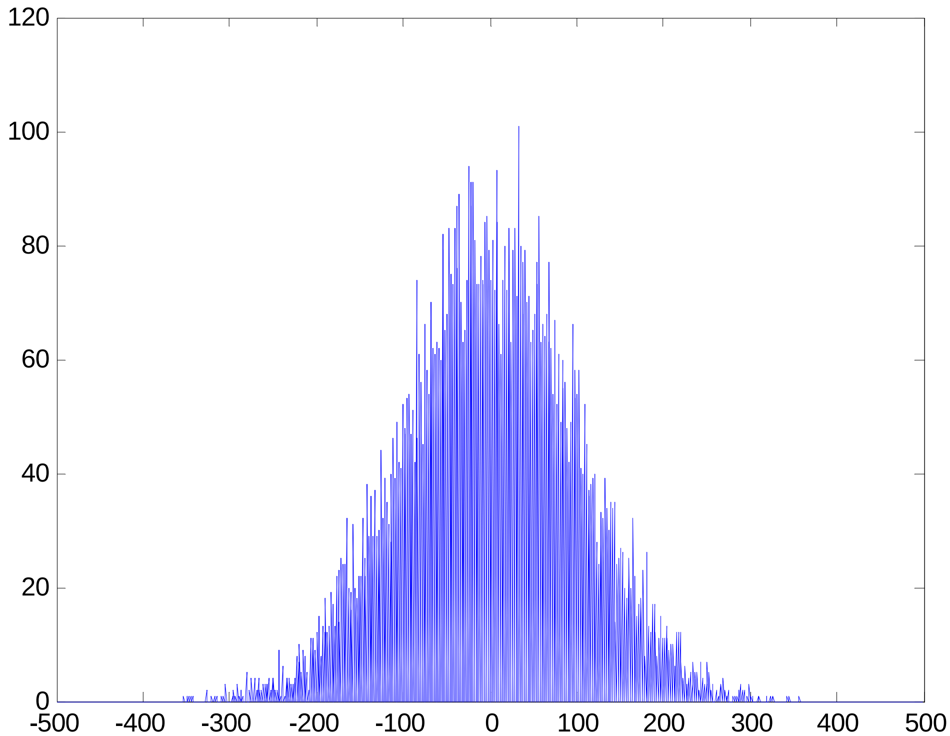
~~Confidence interval~~

$$\cancel{P[-z \le Z \le z] = \int_{-z}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx} = 1-2Q(z)$$

i.e.,

$$\cancel{1-2Q(z) = P[-z \le \frac{M_n-\mu}{\sigma/\sqrt{n}}]}$$

```
for i=1:10000
    number(i)=0;
end
y=-500:1:500;
for i=1:10000
    x(i)=0;
    for j=1:10000
        x(i)=x(i)+sign(0.5-rand());
    end
    number(x(i)+500)=number(x(i)+500)+1;
end
plot(y(1:1000),number(1:1000))
```



Assume $X_1, \cdots X_n$ are i.i.d. Gaussian and we have found a sample mean $M_n = \frac{1}{n}\sum_{i=1}^{n} X_n$ and we would like to know if it is a good measure of ~~real me~~ the actual mean μ or not.

M_n is r.v. which is:

- Gaussian
- has mean μ
- has variance $\frac{\sigma^2}{n}$.

We have

$$1 - 2Q(z) = P\left[-z \leq \frac{M_n - \mu}{\sigma/\sqrt{n}} \leq z\right]$$

$$= P\left[M_n - \frac{z\sigma}{\sqrt{n}} \leq \mu \leq M_n + \frac{z\sigma}{\sqrt{n}}\right]$$

let $2Q(z_{\alpha/2}) = \alpha$.

then with probability $(1-\alpha)$ μ is in the range $M_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $M_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

this is called $(1-\alpha)\%$ confidence interval.

Example :

A voltage $X = v + N$

where v is an unknown constant voltage and N is a random noise voltage: Gaussian with zero mean and variance $1\,\mu V^2$.

Find the 95% confidence interval for v if the voltage X is measured 100 times and the sample mean found is $5.25\,\mu V$.

$$2Q(z_{\alpha/2}) = \alpha = 0.05$$

$$Q(z_{\alpha/2}) = 0.025 \Rightarrow z_{\alpha/2} = 1.96$$

So, the 95% confidence interval is:

$$\left(5.25 - \frac{1.96 \times \overset{\sigma}{1}}{\underset{\sqrt{n}}{10}},\ 5.25 + \frac{1.96 \times 1}{10}\right)$$

or:

$$(5.05, 5.45)\ \mu V$$

Random Processes

Assume that we have a random experiment with outcomes ω belonging to the sample set S.

to each $\omega \in S$ we assign a time function:

$$X(t, \omega) \qquad t \in I$$

where I is a time <u>index</u> <u>set</u> : discrete or Continuous depending on the case.

<u>example</u>:

A random experiment with two outcomes $\omega \in \{0, 1\}$

and

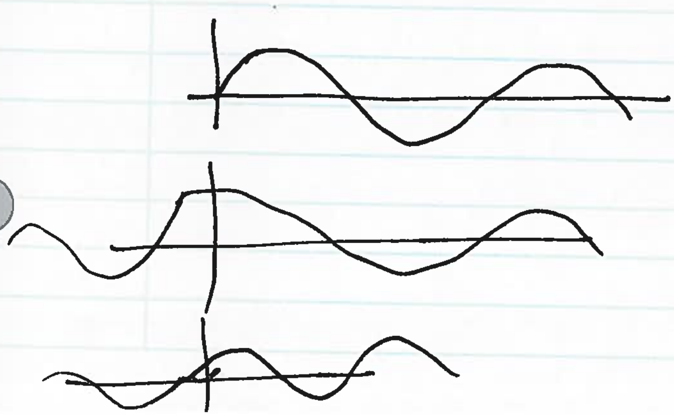
$$X(t,0) = A \cos \omega t \qquad t \in [0,T] \text{ or } t \in [kT, (k+1)T$$

$$X(t,1) = A \sin \omega t \qquad t \in [0,T]$$

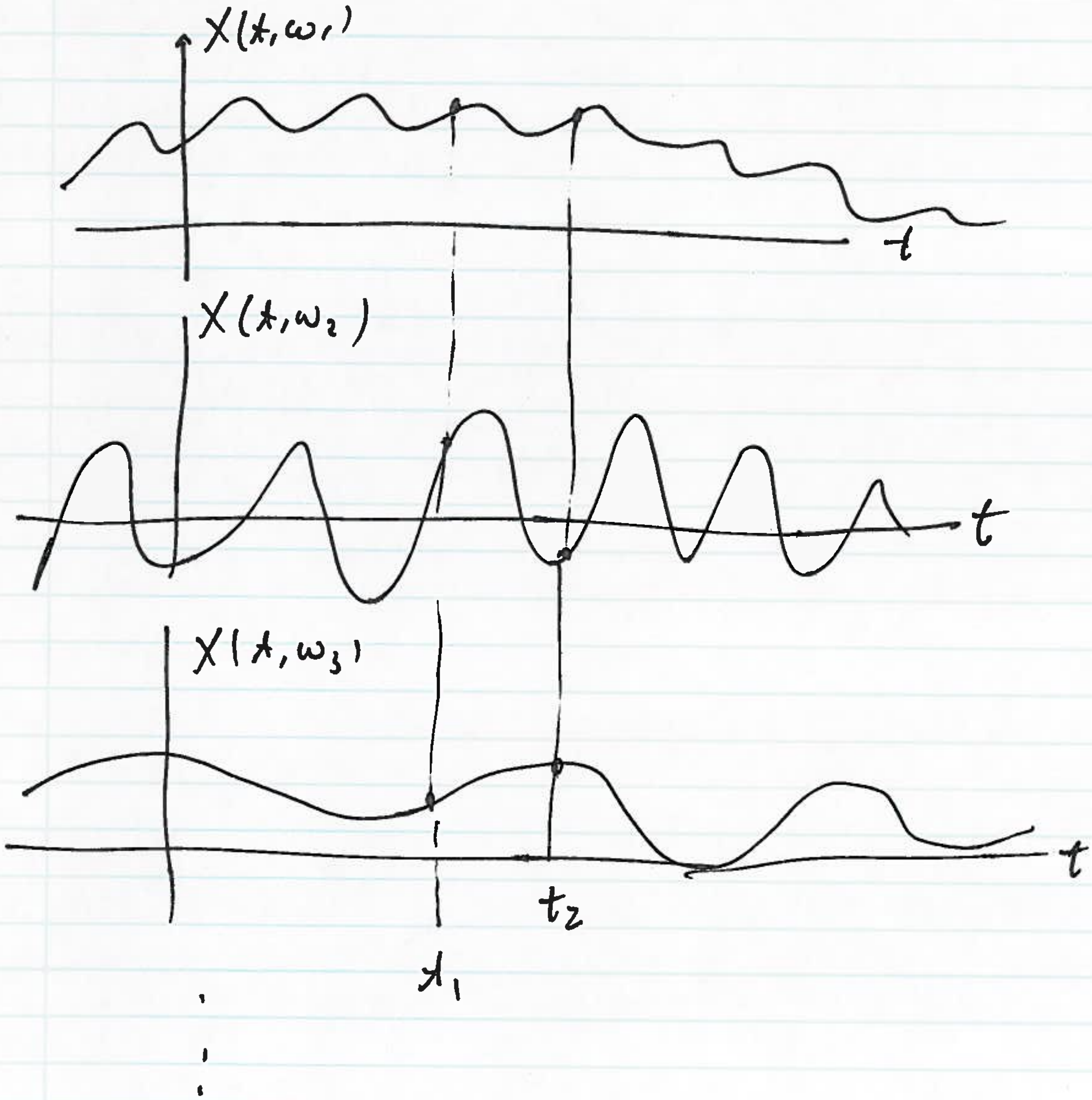
<u>example</u>:

random experimen Consisting of choosing a real number in the range of $[0, 2\pi]$ uniformly and

$$X(t, \varphi) = A \cos(\omega t + \varphi)$$



Usually, we drop ω and show the random process as $X(t)$.



at any given time t the set $\{X(t, \omega)\}$ is a random variable. So, a random process consists of a set of time-indexed random variables:

$$\{X(t, \omega), \ t \in I\}$$