X Lecture 9, Nov. 3, 04

Specifying a random process

Join Distribution of Time Samples:

Let $X_1, X_2, \cdots X_n$ be the samples of $X(t, \omega)$ obtained at $t_1, t_2, \cdots t_n$, i.e.,

$$X_i = X(t_i, \omega) \quad i \in \{1, 2, \cdots n\}$$

we define

$$F_{X_1 \cdots X_n}(x_1, x_2, \cdots x_n) = P[X_1 \leq x_1, \cdots X_n \leq x_n]$$

and (in discrete case)

$$p_{X_1 \cdots X_n}(x_1, \cdots x_n) = P[X_1 = x_1, \cdots, X_n = x_n] \qquad \text{pmf}$$

for continuous case, we define (pdf)

$f_{X_1 \cdots X_n}(x_1, \cdots x_n)$ as the derivative of CDF.

Example

Let X_i, $i \in \{1, \cdots, n\}$ be a sequence of i.i.d. Bernouli r.v.'s with $p = \frac{1}{2}$. Then

$$P[X_1 = x_1, \cdots X_n = x_n] = 2^{-n} \quad \text{all } (x_1, \cdots x_n) \in \{0, 1\}^n$$

Mean, Autocorrelation and Auto covariance Functions

mean function:

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x \, f_{X(t)}(x)\, dx$$

Autocorrelation function:

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{X(t_1)X(t_2)}(x,y)\, dx\, dy$$

Auto covariance function

$$C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$

It is easy to show that

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

$$Var[X(t)] = E[(X(t) - m_X(t))^2] = C_X(t, t)$$

The correlation ~~function~~ coefficent

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)\, C_X(t_2, t_2)}}$$

mean and autocorrelation function provide a partial description of a random process. Only in certain cases (Gaussian), they can ~~be~~ provide a full description.

Example:

$X(t) = A \cos(2\pi t)$ where A is a random variable.

$$m_X(t) = E[A \cos 2\pi t] = E[A] \cos 2\pi t$$

and

$$R_X(t_1, t_2) = E[(A \cos 2\pi t_1)(A \cos 2\pi t_2)]$$
$$= E[A^2] \cos 2\pi t_1 \cos 2\pi t_2$$

and

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$
$$= \left(E[A^2] - E^2[A]\right) \cos 2\pi t_1 \cos 2\pi t_2$$
$$= Var(A) \cos 2\pi t_1 \cos 2\pi t_2$$

Example: $X(t) = A \cos(\omega t + \theta)$ where θ is uniform on $[0, 2\pi]$

$$m_X(t) = E[A \cos(\omega t + \theta)] = \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega t + \theta)\, d\theta = 0$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) = A^2 E[\cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta)]$$
$$= \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{2}\{[\cos \omega(t_1 - t_2)] + \cos[\omega(t_1 + t_2) + \theta]\}\, d\theta$$
$$= \frac{A^2}{2} \cos \omega(t_1 - t_2)$$

Gaussian Random Processes

A random process $X(t)$ is a **Gaussian Random Process** if for any n, the samples taken at $t_1, t_2, \ldots, t_n$ are jointly Gaussian, i.e., if

$$X_1 = X(t_1) \; \cdots \; X_n = X(t_n)$$

then

$$f_{X_1,\ldots X_n}(x_1,\ldots x_n) = \frac{e^{-\frac{1}{2}(\underline{x}-\underline{m})^T K^{-1}(\underline{x}-\underline{m})}}{(2\pi)^{n/2}\,|K|^{1/2}}$$

where

$$\underline{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ \vdots \\ m_X(t_n) \end{bmatrix}$$

and

$$K = \begin{bmatrix} C_X(t_1,t_1) & C_X(t_1,t_2) & \cdots & C_X(t_1,t_n) \\ C_X(t_2,t_1) & C_X(t_2,t_2) & \cdots & C_X(t_2,t_n) \\ \vdots & \vdots & & \vdots \\ C_X(t_n,t_1) & C_X(t_n,t_2) & \cdots & C_X(t_n,t_n) \end{bmatrix}$$

Example:

let the discrete time random process X_n be a sequence of independent Gaussian random variables with mean m and variance σ^2.

Then

$$C_X(t_i, t_j) = \sigma^2 \delta_{ij}$$

i.e.,

$$K = \sigma^2 I = \sigma^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \\ \vdots & \cdots & \cdots & 1 \end{bmatrix}$$

and

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - m)^2}$$

$$= f_X(x_1) f_X(x_2) \cdots f_X(x_n)$$

Multiple random processes

To specify joint random processes $X(t)$ and $Y(t)$, we need to (in general case) have the pdf of all samples of $X(t)$ and $Y(t)$ such as $X(t_1) \dots X(t_i), Y(t'_1) \dots Y(t'_j)$ for all i and j and all choices of $t_1, \dots t_i$ and $t'_1, \dots t'_j$.

The processes $X(t)$ and $Y(t)$ are independent if the random vectors

$(X(t_1), \ldots X(t_i))$ and $(Y(t_1'), \ldots Y(t_j'))$

are independent for all ~~ks~~ i, j and all choices of $t_1, \ldots t_i$ and $t_1', \ldots t_j'$.

The <u>cross-correlation</u> $R_{X,Y}(t_1, t_2)$ is defined as:

$$R_{X,Y}(t_1, t_2) = E[X(t_1) X(t_2)]$$

Two processes are <u>orthogonal</u> if

$$R_{X,Y}(t_1, t_2) = 0 \text{ for all } t_1 \text{ and } t_2 .$$

The <u>cross Covariance</u>

$$C_{X,Y}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}]$$
$$= R_{X,Y}(t_1, t_2) - m_X(t_1) m_X(t_2).$$

Two processes X and Y are said to be <u>uncorrelated</u> if

$$C_{X,Y}(t_1, t_2) = 0 \quad \text{all } t_1 \text{ \& } t_2 .$$

Example:

$$X(t) = \cos(\omega t + \theta) \quad \& \quad Y(t) = \sin(\omega t + \theta)$$

θ is uniform on $[0, 2\pi]$

$$R_{X,Y}(t_1, t_2) = E[\cos(\omega t_1 + \theta)\sin(\omega_2 t + \theta)]$$

$$= E[-\tfrac{1}{2}\sin\omega(t_1 - t_2) + \tfrac{1}{2}\sin(\omega(t_1+t_2)+2\theta)]$$

$$= -\tfrac{1}{2}\sin\omega(t_1 - t_2)$$

Discrete-Time Random Processes.

i.i.d. processes:

$$X_n \sim f_X(x_n)$$

then

$$F_{X_1,\cdots X_n}(x_1, \cdots x_n) = P[X_1 \le x_1, X_2 \le x_2, \cdots X_n \le x_n]$$

$$= F_X(x_1)F_X(x_2)\cdots F_X(x_n)$$

and the mean is:

$$m_X(n) = E[X_n] = m \qquad \text{all } n$$

$$C_X(n_1, n_2) = E[(X_{n_1} - m)(X_{n_2} - m)]$$

$$= E[X_{n_1} - m]\,E[X_{n_2} - m] = 0 \quad \text{if } n_1 \ne n_2$$

and

$$C_X(n,n) = E[(X_n - m)^2] = \sigma^2$$

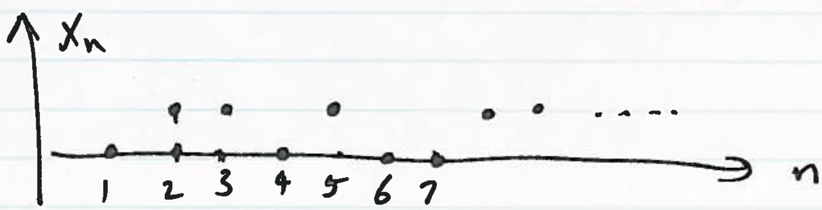
So

$$C_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2}$$

and

$$R_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2} + m^2$$

<u>Example</u>: Let X_n be a sequence of i.i.d. Bernouli random variables; A realization of X_n will be



$$m_X(n) = p \times 1 + (1-p) \times 0 = p$$

$$var(X_n) = p - p^2 = p(1-p)$$

$$P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n] = p^{\sum_{i=1}^{n} x_i} (1-p)^{n - \sum_{i=1}^{n} x_n}$$

e.g.

$$P[X_1 = 1, X_2 = 1, X_3 = 0] = p^2 (1-p)$$

Example:

$Y_n = 2X_n - 1$ where X_n are i.i.d. Bernoulli

then $Y_n = \begin{cases} 1 & \text{with prob. } p \\ -1 & \text{with prob. } (1-p) \end{cases}$

$m_Y(n) = E[Y_n] = p - (1-p) = 2p - 1$

$Var(Y_n) = Var(2X_n - 1) = 4p(1-p)$

Example: Random walk

Take $S_n = Y_1 + Y_2 + \cdots + Y_n$

where Y_n are i.i.d. $\in \{-1, 1\}$ with $P(1) = p$.

This is a one-dimensional random walk

- if there are k positive jumps ($+1$'s) in n trials (n walks), then there are $n-k$ negative jumps (-1's). So

$$S_n = k \times 1 + (n-k) \times (-1) = 2k - n$$

$$\boxed{P[S_n = 2k-n] = \binom{n}{k} p^k (1-p)^{n-k}}$$

Properties of Random walk Process:

independent increment in non-overlapping intervals.

Let $\overbrace{n_0 < n \le n_1}^{I_1}$ and $\overbrace{n_2 < n \le n_3}^{I_2}$

where $n_1 < n_2$, i.e., I_1 & I_2 do not overlap.

$$S_{n_1} - S_{n_0} = Y_{n_0+1} + \cdots + Y_{n+1}$$

$$S_{n_3} - S_{n_2} = Y_{n_2+1} + \cdots + Y_{n_3}$$

Since the two above increments have no Y_n's in common and X_n's are independent, then the two increments are independent, i.e., $S_{n_1} - S_{n_0}$ and $S_{n_3} - S_{n_2}$ are two independent random variables.

Furthermore since $S_{n'} - S_n$ is the sum of $n'-n$ i.i.d. random variables $Y_{n+1}, \dots Y_{n'}$, it has the same distribution as $S_{n'-n}$, i.e., the sum of $n'-n$ i.i.d., Y's. That is,

$$P[S_{n'} - S_n = s] = P[S_{n'-n} = s]$$

This means that the increments over intervals of the same length have the same distribution. So, the process S_n is ~~[illegible]~~ said to have <u>stationary increments</u>.

This fact can be used to <u>easily</u> find the joint pmf of S_n at $n_1, n_2 \cdots n_k$.

$$P[S_{n_1}=s_1, S_{n_2}=s_2, \cdots S_{n_k}=s_k] = P[S_{n_1}=s_1]P[S_{n_2}-S_{n_1}=s_2-s_1]$$
$$\times P[S_{n_3}-S_{n_2}=s_3-s_2] \cdots$$
$$\times P[S_{n_k}-S_{n_{k-1}}=s_k-s_{k-1}]$$
$$= P[S_{n_1}=s_1]\, P[S_{n_2-n_1}=s_2-s_1] \cdots P[S_{n_k-n_{k-1}}=s_k-s_{k-1}]$$

If Y_n are continuous valued random variables,

$$f_{S_{n_1},\cdots S_{n_k}}(s_1,\cdots s_k) = f_{S_{n_1}}(s_1)\, f_{S_{n_1-n_2}}(s_2) \cdots f_{S_{n_k-n_{k-1}}}(s_k-s_{k-1})$$

Let the increment Y_n be $\sim N(0, \sigma^2)$

then

$$f_{S_{n_1},S_{n_2}}(s_1,s_2) = f_{S_{n_1}}(s_1)\, f_{S_{n_2-n_1}}(s_2-s_1)$$
$$= \frac{1}{\sqrt{2\pi n_1}\,\sigma} e^{-s_1^2/2n_1\sigma^2} \cdot \frac{1}{\sqrt{2\pi(n_2-n_1)}\,\sigma} e^{\frac{-(s_2-s_1)^2}{2\sigma^2(n_2-n_1)}}$$