The Gradient and Applications

- This unit is based on Sections 9.5 and 9.6, Chapter 9.
- All assigned readings and exercises are from the textbook

Objectives:
Make certain that you can define, and use in context, the terms, concepts and formulas listed below:
1. find the gradient vector at a given point of a function.
2. understand the physical interpretation of the gradient.
3. find a multi-variable function, given its gradient
4. find a unit vector in the direction in which the rate of change is greatest and least, given a function and a point on the function.
5. the rate of change of a function in the direction of a vector.
6. find normal vector and tangent vectors to a curve
7. write equations for the tangent line and the normal line.
8. find an equation for the tangent plane to a surface

Reading: Read Section 9.5, pages 474-482.
Exercises: Complete problems 9.5 and 9.6
Prerequisites: Before starting this Section you should . . .
- familiar with the concept of partial differentiation
- be familiar with vector functions
Directional Derivatives: definition

- Consider the temperature $T$ at various points of a heated metal plate.
- Some contours for $T$ are shown in the diagram.

- We are interested in how $T$ changes from one point to another.
Directional Derivatives: definition

- The rate of change of $T$ in the direction specified by $AB$ is given by:
  \[(20-15)/AB = 5/h\]
  an example of directional derivative

- In general, for a given function $T = T(x,y)$, the directional derivative in the direction of a unit vector $u = \langle \cos \theta, \sin \theta \rangle$ is

\[
D_u(T) = \lim_{h \to 0} \frac{T(x + h \cos \theta, y + h \sin \theta) - T(x, y)}{h},
\]
where \[h = \sqrt{(\Delta x)^2 + (\Delta y)^2}\]
Directional Derivatives: definition

- The **gradient vector** at a point $A$
  - magnitude = the largest directional derivative, and
  - pointing in the direction in which this largest directional derivative occurs, is known as the gradient vector.
The gradient vector: \( \text{grad} \)

- A vector field, called the gradient, written: \( \text{grad} \ F \) or \( \nabla F \)
can be associated with a scalar field \( F \).
- At every point the direction of the vector field (\( \nabla F \)) is
  - orthogonal to the scalar field contour (\( C \)) and
  - in the direction of the maximum rate of change of \( F \).

\[
\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}
\]

\( \nabla \) called ‘\( \text{del} \)’
The gradient vector: grad

- Gradient of a Function

**Given:** \( w = F(x, y, z) \)

\[ \therefore \text{grad } F = \nabla F(x, y, z) = i \frac{\partial F}{\partial x} + j \frac{\partial F}{\partial y} + k \frac{\partial F}{\partial z} \]

**Example:**

**Given:** \( F = \frac{xy^2}{z^3} \), **Find** \( \nabla F \)
The gradient vector (Cont.)

Key points:

- $\nabla F$ direction is the normal vector to the surface
  \[ \nabla F \cdot \hat{t} = 0, \quad \hat{t} \text{ is a tangent unit vector to } F \]

- $|\nabla F|$ magnitude gives the rate of change (the slope) (rate of change) of $F$, when moving along a certain direction.

- $\nabla F$ points in the direction of most rapid increase of $F$.

- $-\nabla F$ points in the direction of most rapid decrease of $F$.

- $F$ is a scalar field while $\nabla F$ is a vector field.

- $\nabla F$ is not constant in space.
The gradient vector (Cont.)

- Generalization of **Directional Derivative** in \( u \) direction:

\[
D_u(F) = \nabla F \cdot \hat{u}, \quad \hat{u} = \text{unit vector}
\]

- The **maximum** value of \( D_u(F) \) is \( ||\nabla F|| \) and occurs when \( u \) and \( \nabla F \) are in the **same** directions.

- The **minimum** value of \( D_u(F) \) is \(-||\nabla F||\) and occurs when \( u \) and \( \nabla F \) are in the **opposite** directions.

**Example**: \( F = xy/(x + y); \quad \vec{u} = \langle 6, 8 \rangle \quad \text{Find } D_u(F) \text{ @ } (2, -1) \).
The gradient vector: Some Applications

- Engineers use the gradient vector in many physical laws such as:

1. Electric Field ($\mathbf{E}$) and Electric Potential ($V$):
   $$\mathbf{E}(x, y, z) = -\nabla V(x, y, z)$$

2. Heat Flow ($\mathbf{H}$) and Temperature ($T$):
   $$\mathbf{H}(x, y, z) = k\nabla T(x, y, z), \quad k = \text{constant}$$

3. Force Field ($\mathbf{F}$) and Potential Energy ($U$):
   $$\mathbf{F}(x, y, z) = -\nabla U(x, y, z)$$

**Example**

Given: $T = 100 - 2x^2 - y$

Find the heat flow vector $\mathbf{H}$
Insect Example!

- **Temperature Distribution:**

\[ T(x, y) = 5 + 2x^2 + y^2 \]

Coolest location: center

What direction the insect should go to cool off the fastest?

(Assuming the insect is familiar with vector analysis and knows the temperature distribution!!)

\[ \mathbf{d} = -\nabla T(x, y) = -4xi - 2yj \]

\[ = -16i - 4j \]
Equation of Tangent Plane

- A vector normal to the surface \( F(x,y,z) = c \) at a point \( P (x_o,y_o,z_o) \) is \( \nabla F \), and can be denoted by \( n_o \).

- If \( r_o \) is the position vector of the point \( P \) relative to the origin, and \( r \) is the position vector of any point on the tangent plane, the vector equation of the tangent plane is:

\[
\vec{n}_o \cdot (\vec{r} - \vec{r}_o) = 0, \quad \vec{n}_o = \nabla F(r_o) \text{ at } P
\]

- The equivalent scalar equation of the tangent plane is:

\[
(x-x_o)F_x(x_o,y_o,z_o)+(y-y_o)F_y(x_o,y_o,z_o)+(z-z_o)F_z(x_o,y_o,z_o)=0
\]

**Example:** Surface: \( x^2 + y^2 + z^2 = 9 \), Point: \((-2,2,1)\)
Equation of Normal Line to a Surface

- A vector normal to the surface \( F(x,y,z) = c \) at a point \( P(x_0,y_0,z_0) \) is \( \mathbf{n}_o = \nabla F \).

- If \( \mathbf{r}_o \) is the position vector of the point \( P \), and \( \mathbf{r} \) is the position vector of any point on the normal line, the vector equation of the normal line to the surface is:

\[
\mathbf{n}_o \times (\mathbf{r} - \mathbf{r}_o) = 0, \quad \mathbf{n}_o = \nabla F(\mathbf{r}_o) \text{ at } P
\]

- The equivalent parametric equation of the normal line is:

\[
\begin{align*}
x &= x_0 + t F_x(x_0,y_0,z_0), \\
y &= y_0 + t F_y(x_0,y_0,z_0), \\
\quad z &= z_0 + t F_z(x_0,y_0,z_0)
\end{align*}
\]

**Example:** Surface: \( x^2 + 2y^2 + z^2 = 4 \), Point: \((1, -1, 1)\)