Line Integral and Its Independence of the Path

- This unit is based on Sections 9.8 & 9.9, Chapter 9.
- All assigned readings and exercises are from the textbook.

Objectives:
- Make certain that you can define, and use in context, the terms, concepts and formulas listed below:
  1. integrate a 2D or 3D vector field over a path.
  2. calculate the work done by a 2D or 3D force operating over a path.
  3. dependence and independence of a line integral of the path.
  4. conservative field and potential functions.

Reading: Read Section 9.8 & 9.9, pages.

Exercises: Complete problems.

Prerequisites: Before starting this Section you should...
- ✓ be familiar with the concept of integration.
- ✓ be familiar with vector algebra and partial derivatives.
- ✓ be familiar with 3D coordinate system.
Why line integrals?

- The work $\Delta W$ done by a force $\mathbf{F}$ in the displacement of a particle along a curve $C$ (given by $\mathbf{r}(t)$) a small distance $\Delta d$ is:

  $$\Delta W = \mathbf{F} \cdot \Delta \mathbf{d} = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{F} \cdot \Delta \mathbf{r}$$

  Where: $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_1(x, y, z)\mathbf{k}$

- The total work $W$ done by the force $\mathbf{F}$ is given by the sum:

  $$W = \sum \Delta W = \sum \mathbf{F} \cdot \Delta \mathbf{r}$$

- In the limit when $\Delta \mathbf{r}$ approaches zero, we have:

  $$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_c F_1(x, y, z)dx + \int_c F_2(x, y, z)dy + \int_c F_3(x, y, z)dz$$

- As a function of time $(t)$:

  $$W = \int_C \mathbf{F} \cdot \mathbf{r}'(t)dt = \int_A^B \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t)dt$$
Methods of Evaluation: I

\[ I = \int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x, y, z)dx + \int_C F_2(x, y, z)dy + \int_C F_3(x, y, z)dz \]

\[ \text{C is defined by a parametric equation} \]

\[ C : \ x = f(t); \ y = g(t); \ z = h(t), \ \text{with} \ a \leq t \leq b \]

\[ I_1 = \int_C F_1(x, y, z)dx = \int_a^b F_1(f(t), g(t), h(t)) f'(t) \, dt \]

\[ I_2 = \int_C F_2(x, y, z)dy = \int_a^b F_2(f(t), g(t), h(t)) g'(t) \, dt \]

\[ I_3 = \int_C F_3(x, y, z)dy = \int_a^b F_3(f(t), g(t), h(t)) h'(t) \, dt \]

**Example:** P9.8-15, \( x = \sqrt{t}, y = t, \ 4 \leq t \leq 9 \)

\[ \int_C (6x^2 + 2y^2) \, dx + 4xy \, dy \]

**Example:** P9.8-30 Find the work done by force \( \vec{F} \) along the curve \( \vec{r} \)

\[ W = \int_C \vec{F} \cdot d\vec{r}, \ \vec{F} = \langle e^x, xe^{xy}, xy e^{xyz} \rangle, \ \vec{r} = \langle t, t^2, t^3 \rangle, \ 0 \leq t \leq 1 \]
Methods of Evaluation: II

\[ I = \int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x, y)dx + \int_C F_2(x, y)dy \]

- \( C \) is defined by an explicit function
  
  \[ C: \quad y = f(x); \quad \text{with} \quad a \leq x \leq b \]

\[
I_1 = \int_C F_1(x, y)dx = \int_a^b F_1(x, f(x)) \, dx
\]

\[
I_2 = \int_C F_2(x, y)dy = \int_a^b F_2(x, f(x)) \, f'(x) \, dx
\]

**Example:** P9.8-27,

\[
\int_C y \, dx + z \, dy + x \, dz
\]
Methods of Evaluation: III
Integration with respect to the arc length

1. C is defined by a parametric equation

\[ C : \quad x = f(t); \quad y = g(t); \quad z = h(t), \quad \text{with} \quad a \leq t \leq b \]

\[ I = \int_C F(x, y, z) ds = \int_a^b F(f(t), g(t), h(t)) \sqrt{[f'(t)]^2 + g'^2 + h'^2} \, dt \]

2. C is defined by an explicit function

\[ C : \quad y = f(x); \quad \text{with} \quad a \leq x \leq b \]

\[ I = \int_C F(x, y) ds = \int_a^b F(x, f(x)) \sqrt{1 + [f'(x)]^2} \, dx \]

Example: P9.8-40, Find m (mass of a wire) where \( \rho \) is the density:

\[ m = \int_C \rho \, ds \quad , \quad C: x=1+\cos(t), y=\sin(t), \quad 0 \leq t \leq \pi, \quad \rho = kx \]
Circulation and integration on closed curves

\[ I = \oint_C \vec{F}(x, y) \cdot d\vec{r} \quad \& \quad I = \oint_C F(x, y) ds \]

**Example:** P9.8-21, Evaluate

\[ \oint_C x^2 y^3 \, dx - x y^2 \, dy \]
Dependence and Independence on the Path

- Consider the line integral (2D case):
  \[ I = \int_C \vec{F}(x, y) \cdot d\vec{r} = \int_A^B [P(x, y)dx + Q(x, y)dy] \]

The line integral depends in general on the integrand function, the end points of the path (A and B), and on the path C.

- The line integral \( I \) is independent of the path C, \textit{if and only if}:
  \[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

  \textbf{Example: P9.9-3}

- In 3D space, the integral:
  \[ I = \int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_A^B [P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz] \]

  is independent of the path C, \textit{if and only if}:
  \[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \& \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \]

  \textbf{Example: P9.9-21}
Independence on the Path (Cont.)

• To verify that the following integral is independent of the path

\[ I = \int_{A}^{B} [P(x, y)dx + Q(x, y)dy] \]

Show that

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

• To calculate \( I \), find a function \( \phi \) such that

\[ d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \implies \frac{\partial \phi}{\partial x} = P \quad \& \quad \frac{\partial \phi}{\partial y} = Q \]

\[ \therefore I = \int_{A}^{B} d\phi = \phi \bigg|_{A}^{B} = \phi(B) - \phi(A) \]
**Example: P9.9-3** Show that the following integral is independent of the path

\[ I = \int_{(1,0)}^{(3,2)} [(x + 2y)dx + (2x - y)dy] \]

Two different ways:

a) Find \( \phi \) such that \( d\phi = P \, dx + Q \, dy \)

b) Integrate along a convenient path

**Example: P9.9-21** Show that the following integral is independent of the path. Evaluate the integral

\[ I = \int_{(1,0,0)}^{(3,\pi/2,1)} \left[ (2x \sin y + e^{3z})dx + x^2 \cos y \, dy \right. \]

\[ \left. + (3xe^{3z} + 5)dz \right] \]

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \& \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \]
Conservative Fields and Potential Functions

For a conservative field \( \mathbf{F} \), the following are applied:

\[
\nabla \times \mathbf{F} = 0
\]

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{s} = 0; \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
\]

\[
\int_{A}^{B} \mathbf{F} \cdot d\mathbf{s} \quad \text{is independent of the path between A and B}
\]

It can also be shown that any **conservative field** (\( \mathbf{F} \)) can be expressed as the gradient of some **scalar function** \( \phi \), known as a **potential function**

\[
\mathbf{F} = \nabla \phi = P(x, y)\hat{i} + Q(x, y)\hat{j}
\]
Example:

Show that the following vector field is conservative and find a corresponding potential function $\phi$

$$\vec{F} = \langle 3x^2y^2, 2x^3y \rangle$$

Example:

Show that the following scalar field is a potential field (or gradient field) and find the corresponding conservative field

$$\phi = 2xy + zx$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \text{&} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$