Divergence and Curl of a Vector Function

- This unit is based on Section 9.7, Chapter 9.
- All assigned readings and exercises are from the textbook

Objectives:
Make certain that you can define, and use in context, the terms, concepts and formulas listed below:
1. find the divergence and curl of a vector field.
2. understand the physical interpretations of the Divergence and Curl.
3. solve practical problems using the curl and divergence.

- Reading: Read Section 9.7, pages 483-487.
- Exercises: Complete problems

Prerequisites: Before starting this Section you should . . .
- be familiar with the concept of partial differentiation
- be familiar with vector dot and cross multiplications
- be familiar with 3D coordinate system
Differentiation of vector fields

- **Example of a vector field**: Suppose fluid moves down a pipe, a river flows, or the air circulates in a certain pattern. The velocity can be different at different points and may be at different time.

- The velocity vector $\mathbf{F}$ gives the direction of flow and speed of flow at every point.

- **Applications of Vector Fields**:
  - Mechanics
  - Electric and Magnetic fields
  - Fluids motions
  - Heat transfer

- There are two kinds of differentiation of a vector field $\mathbf{F}(x,y,z)$:
  1. divergence ($\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$) and
  2. curl ($\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$)
Examples of Vector Fields

(a) Airflow around an airplane wing

(b) Laminar flow of blood in an artery; cylindrical layers of blood flow faster near the center of the artery

(d) Lines of force around two equal positive charges
The Divergence of a Vector Field

- Consider the vector fields

  Vector function with two variable:
  $$\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$$

  Vector function with three variable:
  $$\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$$

- We define the divergence of $\vec{F}$

  $$\text{Div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- In terms of the differential operator $\nabla$, the divergence of $\vec{F}$

  $$\text{Div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- **A key point**: $\vec{F}$ is a vector and the divergence of $\vec{F}$ is a scalar.

**Example**: $\vec{F} = 4xy\hat{i} + (2x^2 + 2yz)\hat{j} + 3(z^2 + y^2)\hat{k}$, **Find** $\nabla \cdot \vec{F}$
Divergence

- Divergence is the outflow of flux from a small closed surface area (per unit volume) as volume shrinks to zero.

- Air leaving a punctured tire: Divergence is positive, as closed surface (tire) exhibits net outflow.

- The divergence measures sources and drains of flow:

\[
\nabla \cdot \mathbf{F}(\mathbf{P}) > 0 \Rightarrow \text{source}
\]

\[
\nabla \cdot \mathbf{F}(\mathbf{P}) < 0 \Rightarrow \text{sink}
\]

\[
\nabla \cdot \mathbf{F}(\mathbf{P}) = 0 \Rightarrow \text{no source or sink}
\]
Physical Interpretation of the Divergence

- Consider a vector field $\mathbf{F}$ that represents a fluid velocity:
  The divergence of $\mathbf{F}$ at a point in a fluid is a measure of the rate at which the fluid is flowing away from or towards that point.

- A positive divergence is indicating a flow away from the point.

- Physically divergence means that either the fluid is expanding or that fluid is being supplied by a source external to the field.

- The lines of flow diverge from a source and converge to a sink.

- If there is no gain or loss of fluid anywhere then $\text{div } \mathbf{F} = 0$. Such a vector field is said to be solenoidal.

- The divergence also enters electrical engineering topics such as electric and magnetic fields:
  - For a magnetic field: $\nabla \cdot \mathbf{B} = 0$, that is there are no sources or sinks of magnetic field, a solenoidal filed.
  - For an electric field: $\nabla \cdot \mathbf{E} = \rho/\varepsilon$, that is there are sources of electric field..
The Curl of a Vector Field

- Consider the vector fields
  \[ \vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k} \]
- The curl of \( \vec{F} \) is another vector field defined as:
  \[
  \text{curl} \ \vec{F} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  P & Q & R 
  \end{vmatrix}
  \]
- In terms of the differential operator \( \nabla \), the curl of \( \vec{F} \)
  \[
  \text{Curl} \ \vec{F} = \nabla \times \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\hat{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\hat{k}
  \]
- **A key point**: \( \vec{F} \) is a vector and the curl of \( \vec{F} \) is a vector.

**Example**: \( \vec{F} = 4xy\hat{i} + (2x^2 + 2yz)\hat{j} + 3(z^2 + y^2)\hat{k} \), Find \( \nabla \times \vec{F} \).
Physical Interpretation of the Curl

- Consider a vector field \( \mathbf{F} \) that represents a fluid velocity:
  The curl of \( \mathbf{F} \) at a point in a fluid is a measure of the rotation of the fluid.
- If there is no rotation of fluid anywhere then \( \nabla \times \mathbf{F} = 0 \). Such a vector field is said to be irrotational or **conservative**.
- For a 2D flow with \( \mathbf{F} \) represents the fluid velocity, \( \nabla \times \mathbf{F} \) is **perpendicular** to the motion and represents the direction of axis of rotation.

\[ \begin{align*}
\text{Related Course:} & \quad \text{ENGR361} \\
\text{(a) Irrotational flow} & \quad \text{(b) Rotational flow}
\end{align*} \]

- The curl also enters electrical engineering topics such as electric and magnetic fields:
  - A magnetic field (denoted by \( \mathbf{H} \)) has the property \( \nabla \times \mathbf{H} = \mathbf{J} \).
  - An electrostatic field (denoted by \( \mathbf{E} \)) has the property \( \nabla \times \mathbf{E} = 0 \), an irrotational (conservative) field. **Related Course:** Elec 251/351
Further properties of the vector differential operator $\nabla$

1) $\text{div[grad } f(x, y, z)\text{]} = \nabla \cdot \nabla f = \nabla^2 f$;

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$\nabla^2$ is called the Laplacian operator

2) $\nabla[f(r)g(r)] = g\nabla f + f\nabla g$

3) $\nabla \cdot [f(r)\vec{F}(r)] = f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f$

4) $\nabla \times [f(r)\vec{F}(r)] = f\nabla \times \vec{F} + (\nabla f) \times \vec{F}$

5) $\nabla \cdot [\vec{F}(r) \times \vec{G}(r)] = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

6) $\text{div[curl } \vec{F}(r)\text{]} = \nabla \cdot (\nabla \times \vec{F}) = 0$

7) $\text{curl[grad } f(r)\text{]} = \nabla \times (\nabla f) = 0$

**Verification Examples:**

$f = x^2 y^2 z^3$; \quad $\vec{F} = < x^2 y, xy^2 z, -yz^2 >$
Vector Calculus and Heat Transfer

- Consider a solid material with **density** $\rho$, **heat capacity** $c$, the **temperature distribution** $T(x,y,z,t)$ and **heat flux vector** $\mathbf{q}$.

- **conservation of heat energy**

$$\frac{\partial}{\partial t}(\rho c T) + \nabla \cdot \mathbf{q} = 0$$

- In many cases the heat flux is given by Fick’s law

$$\mathbf{q} = -k\nabla T$$

- Which results in heat equation:

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T,$$

- **Related Course:** MECH352
Conservation of Mass:

Let $\rho$ be the fluid density and $\mathbf{v}$ be the fluid velocity.

Conservation of mass in a volume gives

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Which can be written as

\[
\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0
\]

Related Course: ENGR361
Vector Calculus and Electromagnetics

- **Maxwell equations in free space**
  - Maxwell Equations describe the transmission of information (internet data, TV/radio program, phone,…) using wireless communication.

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon_0}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- Solutions of this equations are essential for the analysis, design and advancement of wireless devices and system, high-speed electronics, microwave imaging, remote sensing, …etc.

- **Related Courses**: ELEC251, ELEC351, ELEC353, ELEC453, ELEC 456, ELEC 457
Magneto-static Field Example

Magneto-static Field is an example of rotational field

\[ \nabla \times \mathbf{B} = \mathbf{J} \]

\[ \nabla \times \mathbf{B} = 0, \text{ outside the cable} \]

\[ \nabla \times \mathbf{B} \neq 0, \text{ inside the cable} \]