The Non-existence of Finite Projective Planes of Order 10

C. W. H. Lam *, L. Thiel, and S. Swiercz

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Dedicated to the memory of Herbert J. Ryser

Abstract

This note reports the result of a computer search which shows that no 19-point configuration can be extended to a complete finite projective plane of order 10. Previous work has shown that such a plane, if it exists, must contain 24,675 19-point configurations. Together, these results imply the non-existence of a projective plane of order 10. This note presents a brief summary of the previous work and gives some details of the methodology which are not covered by earlier publications. It also argues that, even when the possibility of undetected software or hardware errors is taken into account, the probability is very small that an undiscovered plane of order 10 is missed by all the computer searches.

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1 Introduction

A *finite projective plane of order* n, with $n > 0$, is a collection of n^2+n+1 lines and n^2+n+1 points such that

1. every line contains $n+1$ points,
2. every point is on $n+1$ lines,
3. any two distinct lines intersect at exactly one point, and

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4. any two distinct points lie on exactly one line.

It is known that a plane of order n exists if n is a prime power. The first value of n which is not a prime power is 6. Tarry [18] proved in 1900 that a pair of orthogonal latin squares of order 6 does not exist, which by Bose's 1938 result [3] implies that a projective plane of order 6 does not exist. The celebrated Bruck-Ryser theorem [4] provided another explanation of the non-existence of the plane of order 6. The next open value is $n = 10$. This note reports the result of a computer search for 19-point configurations, which, when taken together with previous results, implies that a plane of order 10 does not exist. This is the first example which shows the necessary condition of the Bruck-Ryser theorem is not sufficient.

In Section 2, we give the basic definitions and a summary of the previous work that has a direct implication on this non-existence result. In Section 3, we present some further details of the latest computer search, together with some of its raw data. In Section 4, we speculate on the possibility of the existence of an undiscovered plane of order 10 that is missed by all the computer searches.

2 Definitions and Summary of Previous Work

One way to represent a projective plane is to use an *incidence matrix* A of size n^2+n+1 by n^2+n+1. The columns represent the points and the rows represent the lines. The entry A_{ij} is 1 if point j is on line i; otherwise, it is 0. In terms of an incidence matrix, the property of being a projective plane is translated into:

1. A has constant row sum $n+1$,
2. A has constant column sum $n+1$,
3. the inner product of any two distinct rows of A is 1, and

4. the inner product of any two distinct columns of A is 1.

In 1970, several researchers [1, 15] started studying the binary error-correcting code associated with a projective plane of order 10. Let A be the incidence matrix of such a plane. Let S be the vector space generated by the rows of A over $\mathbf{F}_2$. A vector in S is called a *codeword*. The *weight* of a codeword is the number of 1's in the codeword. Let w_i be the number of codewords of weight i. We define the *weight enumerator* of S to be

$$\sum_{i=0}^{111} w_i x^i.$$

Each codeword of weight i corresponds to a configuration with i points; please see [8] for more detail. We shall use the terms *codeword of weight* i and i*-point configuration* interchangeably.

In [1], it was noted that the weight enumerator of S is uniquely determined once w_{12}, w_{15}, and w_{16} are known. In [15], MacWilliams, Sloane, and Thompson showed, using a computer, that $w_{15} = 0$. In [5], Bruen and Fisher pointed out that $w_{15} = 0$ follows from an earlier computing result by Denniston [7]. In 1983, we completed a computer search which showed that a codeword of weight 12 cannot be completed to a plane [10]; and so, $w_{12} = 0$.

Carter [6] in 1974 showed that there are six starting 16-point configurations. He completed a computer search for four of the cases and part of the fifth. In 1985, we completed the remaining cases [12]. None of the cases can be completed to a plane; hence, $w_{16} = 0$.

Since w_{12}, w_{15}, and w_{16} are all known, one can compute the weight enumerator, for example, by using the formula in [16]. In particular, if a projective plane of order 10 exists, then it must contain 24,675 codewords of weight 19. Hence, the question of the existence of this projective plane can be settled by searching for 19-point configurations.

Let v be a set of 19 points which form a codeword of weight 19. In [8], Hall showed that there are 6 *heavy* lines each containing 5 points of v, 37 *triple* lines each containing 3 points of v, and 68 *single* lines each containing 1 point of v.

There are 66 different incidence structures of the 6 heavy lines with the 19 points of v. These are our starting cases. In [11], 21 of these cases are eliminated by theoretical arguments. The remaining 45 cases are eliminated by the latest computer search.

3 Methodology and Results

In this section, we describe briefly what the computer programs do and present some of the results. Most of the computing techniques for this search have been covered in [14, 17] and we shall not repeat them.

Each starting case of a 19-point configuration is a 6 by 19 submatrix of the incidence matrix. Next, we find all the possible incidence of these 19 points on the triple lines. These incidence on the heavy and triple lines form a 43 by 19 submatrix, which is called an $A2$. Two $A2$'s are isomorphic if one can be changed into another by independent row and column permutations. Isomorphism testing is performed to keep only the non-isomorphic $A2$'s.

Next, we complete each $A2$ to a 111 by 19 submatrix. By using row permutations, there is only one 111 by 19 submatrix for each $A2$. We then work on the remaining portion of the incidence matrix.

Before we continue the discussion, let us introduce a few new terms. The 19 points (columns) that form the weight 19 codeword are called the *inside* points (columns). The remaining points (columns) are the *outside* points (columns). Each heavy line contains six outside points. The incidence of these 6 points with all the lines forms a *block*.

For each $A2$, we choose one of the heavy lines to define our first block ($B1$). After constructing a $B1$, we choose another heavy line and construct a block 2 ($B2$). Here, things become complicated. If these two heavy lines intersect at an outside point, then block 2 contains only 5 new columns; otherwise, it contains 6. For efficiency purposes, it is much better to choose a pair of heavy lines with an outside intersection. After $B2$, we choose a third heavy line and construct block 3 ($B3$). $B3$ may define either 4, 5 or 6 new columns.

case	A2	B1	B2	B3	B4
3	2,501	49,041	185,786	50,865,248	26
7	13,211	178,152	301,680	102,426,490	51
9	87,807	514,241	1,180,342	372,135,063	187
11	7,397	22,526	446,365	188,555,861	3,335
12	10,966	310,393	610,626	220,590,263	96
13	11,961	281,320	595,027	207,555,111	143
15	43,719	410,258	795,130	257,099,099	130
16	5,958	24,516	377,523	153,970,087	2,862
17	9,110	110,560	184,179	67,576,360	43
18	3,406	112,821	225,932	76,343,970	33
22	18,947	286,239	1,143,150	380,012,245	165
23	8,033	19,273	448,324	167,234,836	4,073
24	17,102	221,488	391,940	135,608,176	90
25	21,180	397,053	1,185,339	394,649,869	167
26	18,970	43,446	1,025,757	407,173,889	6,981
27	1,759	628	66,842	10,784,064	7,527
33	16,509	170,052	377,376	128,933,304	85
34	673	3,331	48,697	6,506,350	5,007
36	5,166	163,255	428,497	132,680,375	60
37	3,215	10,941	190,913	75,742,622	1,282
39	1,010	6,126	60,782	8,947,768	6,252
41	10,102	237,526	648,403	204,246,401	92
42	1,679	9,574	122,205	16,577,023	12,333
43	4,855	20,110	359,588	91,645,137	1,779
46	50	56	4,287	648,628	520
47	912	5,203	80,481	18,595,465	364
48	863	7,743	14,025	2,991,378	40
50	1,064	17,050	84,511	28,227,676	18
51	3,685	7,538	196,506	77,618,882	1,656
53	182	182	70,074	10,792,612	9,505
55	3,656	49,247	138,079	44,283,657	20
56	1,554	5,215	57,551	7,350,159	5,425
58	4,329	14,754	304,865	111,916,027	2,004
60	90	70	6,609	965,031	741
63	196	1,114	19,731	2,349,364	139
65	787	15,514	40,894	7,374,077	151
66	662	4,162	50,324	6,667,582	5,110
Total	343,266	3,730,718	12,468,340	4,177,640,149	78,492

Table 1: Results for cases with outside intersections

case	A2	B1	B2	B3	B4
1	20,129	4,221,458	6,791,584,644	3,168,411,676	365,554
2	11,861	1,927,254	4,569,912,128	2,082,460,720	185,301
5	5,219	1,227,914	1,703,821,550	1,714,000,011	45,195
6	111,538	21,252,105	40,489,180,403	16,906,372,320	1,564,123
8	110,879	20,485,157	39,075,289,979	18,210,863,021	1,659,180
20	27,221	4,877,425	9,609,727,470	5,405,054,514	605,625
21	9,410	1,350,785	3,275,119,750	1,832,805,184	168,613
49	101	14,272	37,047,849	20,058,399	1,813
Total	296,358	55,356,370	105,551,683,773	49,340,025,845	4,595,404

Table 2: Results for cases without outside intersections

Similarly, we extend the incidence matrix to blocks 4 and 5. In general, we choose the blocks to concentrate the outside intersections in the first few blocks.

Of course, there may not be any outside intersections. In fact, there are eight starting cases with no outside intersections, and they are the difficult cases which were handled by the CRAY supercomputer.

Table 1 lists the cases with outside intersections. These cases are easy enough to be handled by a collection of five VAX computers at Concordia. The cases were searched in the order of increasing difficulty. In order to reduce the amount of time required by the last few difficult cases, we used an extra partial isomorphism testing. Let v be a codeword of weight 19 and let s and t be two heavy lines with an outside intersection. Then the new codeword $v + s + t$ is again of weight 19 and its incidence pattern with a new set of 6 heavy lines must be one of the 66 starting cases. If this new pattern corresponds to one which is already done, there is no need to investigate this pattern any further. This method helped to reduce the search for the following cases: 7, 9, 15, 17, 24, 33, 48, and 55. Even with this reduction, the total computing time used at Concordia was the equivalent of 800 days of CPU time on a VAX- 11/780.

Table 2 lists the cases with no outside intersections, which were run on a CRAY-1A and took an estimated 2,000 hours of computing. In both tables, we also list the number of non-isomorphic $A2$'s as well as the counts of the number of times the computer programs successfully completed the incidence matrix to the end of a block. Most of the work was done in extending a completed $B2$ to a $B3$. We estimated that the program investigated 2×10^{14} cases to find the $B3$'s.

After trying out all the cases, we have not found any completion to a full incidence matrix. In fact, we have not found any matrix which completes to block 5. Hence, we have to conclude sadly that a plane of order 10 does not exist.

4 Correctness Speculations

Because of the use of a computer, one should not consider these results as a "proof", in the traditional sense, that a plane of order 10 does not exist. They are experimental results and there is always a possibility of mistakes. Despite all these reservations, we are going to present reasons that the probability of the existence of an undiscovered plane of order 10 is very small.

There are two types of possible mistakes, programming errors and hardware errors. Let us first consider programming errors, which are by far the most common. We use two methods to try to avoid them. Whenever possible, we use two different programs and compare the results. The sixty-six starting configurations are generated by two different programs and are also checked by hand. We also have a simple but slow version of the program that attempts to extend an $A2$ to a complete plane. We run both programs for a few selected $A2$'s and we compared the counts at the block boundaries. We find no discrepancy, which gives us faith in the programs. The second method we employ is to apply internal consistency checking when generating the $A2$'s, which is described in more general terms in [13]. Basically, when performing isomorphism testing, one can predict the number of times that each $A2$ is generated by the program. By keeping track of the count, we can at least check that the program performs as expected. Even with these safeguards, the only definitive statement that we can make is that there is no hint of any errors in our programs. We hope that someone else will do an independent verification of the results. Towards this end, we have kept a careful log of all the searches.

The second type of mistake is an undetected hardware error. The CRAY-1A is reported to have such errors at the rate of about one per one thousand hours of computing. A common error is the random changing of bits in a computer word, which may mean the loss of a branch of the search without us knowing about it. In fact, we did discover one such error. After a hardware failure, we reran the 1,000 $A2$'s just before the malfunction. The counts for the last $A2$ processed before the failure had changed, signalling an undetected hardware error. Yet, if an undiscovered plane of order 10 exists, then it contains 24,675 codewords of weight 19. This means that the plane can be constructed as the extension of 24,675 $A2$'s. If all the 24,675 $A2$'s are isomorphic and an undetected hardware error happens to affect this $A2$, then our answer is wrong. Since there are about half a million $A2$'s, the probability of a random undetected hardware error affecting this special $A2$ is about one in half a million. Since the plane is known to have a trivial collineation group [2, 9, 19], it is more likely that there are at least two non-isomorphic $A2$'s amongst the 24,675 cases. In this case, the probability that they are all affected by undetected hardware errors is infinitesimal.

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Computer Science Department

Concordia University

Montréal Québec Canada H3G 1M8