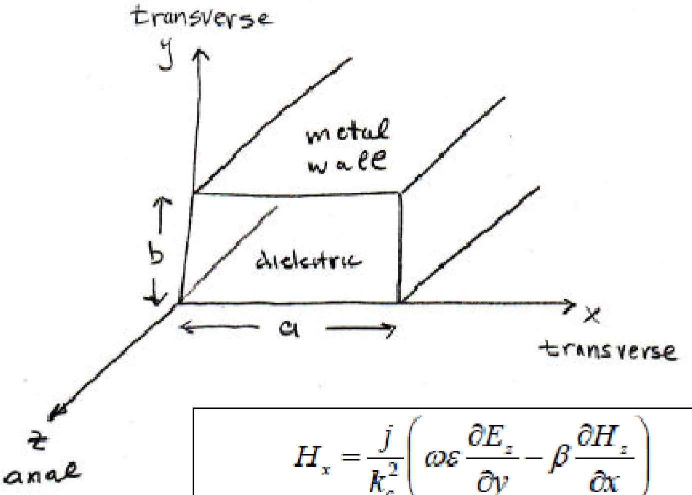
ELEC 351 Notes Set #19

Assignment #4
Ulaby 7.2, 7.3, 7.6, 7.13, 7.14

Transmission Lines and Waveguides

Ulaby Sections 8-6, 8-7, 8-8, 8-9.

Rectangular Waveguides



$k_c^2 = k^2 - \beta^2$

$$\overline{E}(x,y,z) = \overline{e}(x,y)e^{-j\beta z} + \hat{a}_z e_z(x,y)e^{-j\beta z}$$
$$\overline{H}(x,y,z) = \overline{h}(x,y)e^{-j\beta z} + \hat{a}_z h_z(x,y)e^{-j\beta z}$$

the "transverse" field functions are

$$\overline{e}(x,y) = e_x(x,y)\hat{a}_x + e_y(x,y)\hat{a}_y$$
$$\overline{h}(x,y) = h_x(x,y)\hat{a}_x + h_y(x,y)\hat{a}_y$$

the axial components are:

$$E_z = e_z(x,y)e^{-j\beta z}$$
$$H_z = h_z(x,y)e^{-j\beta z}$$

$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$	$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$
$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$	$E_y = \frac{j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$

- "TE modes" having $E_z = 0$
- "TM modes" having $H_z = 0$

TM Modes – Wave Equation

$H_z = 0$

$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial y}\right)$	$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x}\right)$
$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial x}\right)$	$E_y = \frac{j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial y}\right)$

$$\nabla^2 \overline{E} = -k^2 \overline{E}$$

$$\nabla^2 E_z = -k^2 E_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_z = 0$$

$$E_z = e_z(x, y)e^{-j\beta z}$$

$$\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_z}{\partial y^2} - \beta^2 e_z + k^2 e_z = 0$$

$$k_c^2 = k^2 - \beta^2$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)e_z = 0$$

TM Modes – Separation of Variables

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_c^2\right)e_z=0$$

$$e_z(x,y)=X(x)Y(y)$$

$$\frac{\partial^2(XY)}{\partial x^2}+\frac{\partial^2(XY)}{\partial y^2}+k_c^2XY=0$$

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}+k_c^2XY=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+k_c^2=0$$

$$-k_x^2-k_y^2+k_c^2=0$$ Dispersion Equation

$$\frac{1}{X}\frac{d^2X}{dx^2}=-k_x^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_y^2$$

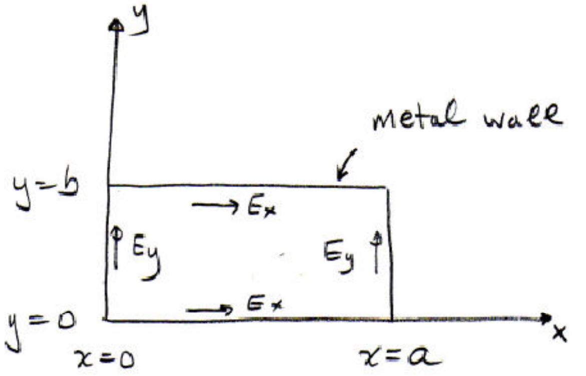
Separation of Variables Solution

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \qquad X(x) = A\cos k_x x + B\sin k_x x$$

$$\frac{d^2Y}{dy^2} + k_y^2 Y = 0 \qquad Y(y) = C\cos k_y y + D\sin k_y y$$

$$e_z(x,y) = (A\cos k_x x + B\sin k_x x)\,(C\cos k_y y + D\sin k_y y)$$

Boundary Conditions



- x=0: $E_z = 0$
- x=a: $E_z = 0$
- y=0: $E_z = 0$
- y=b: $E_z = 0$

Boundary Condition: At x=0, we must have $e_z = 0$, hence

$$e_z(0,y) = (A\cos k_x 0 + B\sin k_x 0)\ (C\cos k_y y + D\sin k_y y) = 0$$

$$e_z(0,y) = (A)\ (C\cos k_y y + D\sin k_y y) = 0$$

which can only be true if $A = 0$ so the field simplifies to

$$e_z(x,y) = (B\sin k_x x)\ (C\cos k_y y + D\sin k_y y)$$

Boundary conditions, continued

Boundary Condition: At y=0, we must have $e_z = 0$ so

$$e_z(x,0) = (B\sin k_x x)\ (C\cos k_y 0 + D\sin k_y 0) = 0$$

$$e_z(x,0) = (B\sin k_x x)\ (C) = 0$$

- This can only be true if $C = 0$ so the field simplifies to

  $$e_z(x,y) = (B\sin k_x x)\ (D\sin k_y y)$$

Boundary Condition: At $x = a$, we must have $e_z = 0$, hence

$$e_z(x = a, y) = (B\sin k_x a)\ (D\sin k_y y) = 0$$

- Aside from the trivial solution of $B = 0$,this can only be true if

  $$k_x a = m\pi \text{ or } k_x = \frac{m\pi}{a}$$

Boundary Condition: At $y = b$, we must have $e_z - 0$, hence

$$e_z(x, y = b) = (B\sin k_x x)\ (D\sin k_y b) = 0$$

- This can only be true if

  $$k_y b = n\pi \text{ or } k_y = \frac{n\pi}{b}$$

Hence the field is… $E_z = B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta_{mn}z}$

$$-k_x^2 - k_y^2 + k_c^2 = 0$$

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$k_c^2 = k^2 - \beta^2 \qquad \beta_{mn} = \sqrt{k^2 - k_c^2}$$

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

TE modes: $H_z = \sum_m \sum_n A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta_{mn}z}$

TM modes: $E_z = \sum_m \sum_n B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta_{mn}z}$

Transverse Field Components

$$E_z = B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta_{mn} z}$$

$H_x = \frac{j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial y}\right)$	$E_x = \frac{j}{k_c^2}\left(\beta \frac{\partial E_z}{\partial x}\right)$
$H_y = \frac{-j}{k_c^2}\left(\omega\varepsilon \frac{\partial E_z}{\partial x}\right)$	$E_y = \frac{j}{k_c^2}\left(\beta \frac{\partial E_z}{\partial y}\right)$

$$E_x = \frac{-j\beta m\pi}{k_c^2 a} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta_{mn} z}$$

$$E_y = \frac{-j\beta n\pi}{k_c^2 b} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta_{mn} z}$$

$$H_x = \frac{j\omega\varepsilon n\pi}{k_c^2 b} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta_{mn} z}$$

$$H_y = \frac{-j\omega\varepsilon n\pi}{k_c^2 a} B_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta_{mn} z}$$

Wave Impedance for TM Modes

- Previously we defined "wave impedance" for TM modes as

$$Z_{TM}=\frac{E_x}{H_y}=\frac{\dfrac{-j}{k_c^2}\beta\dfrac{\partial E_z}{\partial x}}{\dfrac{-j}{k_c^2}\omega\varepsilon\dfrac{\partial E_z}{\partial x}}=\frac{\beta}{\omega\varepsilon}$$

- For mode m,n, the phase constant is $\beta_{mn}=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$, so the wave impedance can be calculated as

$$Z_{TM}=\frac{\beta_{mn}}{\omega\varepsilon}=\frac{\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}{\omega\varepsilon}$$

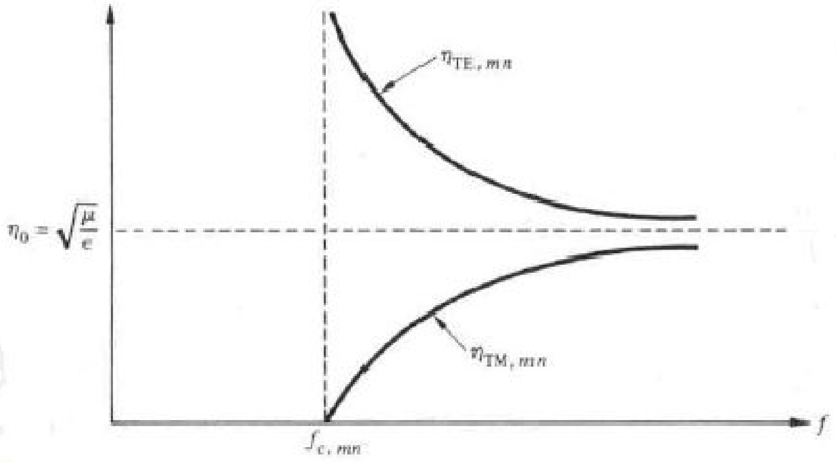
Wave Impedance for TE Modes

$$Z_{TE}=\frac{\omega\mu}{\beta_{mn}}=\frac{\omega\mu}{\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}}$$

Wave Impedance vs. Frequency

FIGURE 8.4

The wave impedances for the modes



From Paul and Nasar, "Introduction to Electromagnetic Fields", McGraw-Hill, 1982, Fig. 8.4

Cutoff Frequencies for TM Modes

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Propagation: β_{mn} must be real

Attenuation: $\beta_{mn} = -j\alpha_{mn}$

To make β_{mn} real we must have

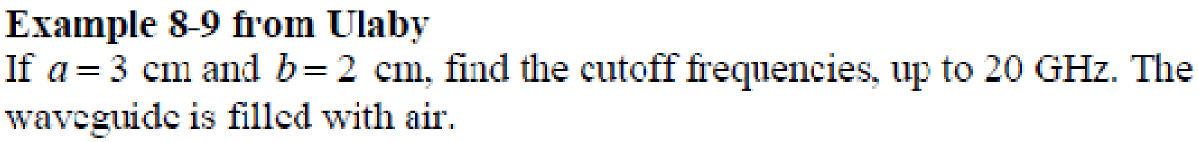
$$k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$k = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu\varepsilon}$

$$f > \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Define the "cutoff frequency" as

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$



Example 8-9 from Ulaby

If $a = 3$ cm and $b = 2$ cm, find the cutoff frequencies, up to 20 GHz. The waveguide is filled with air.

See Ulaby Fig. 8-24

Modal Field Configuration for the TM_{11} Mode

See Ulaby Fig. 8-23