ELEC353 Lecture Notes Set 10

Mid-term Test: Friday February 17, 2012

The mid-term test covers homework assignments 1, 2, 3, 4 and 5.
The mid-term test covers the “self-learning” topic of A.C. Circuit Analysis.

The homework assignments are posted on the course web site.
Homework #4: You should finish this assignment by February 4.
Homework #5: This assignment has **practice problems** for the midterm test. Do this assignment before February 11.

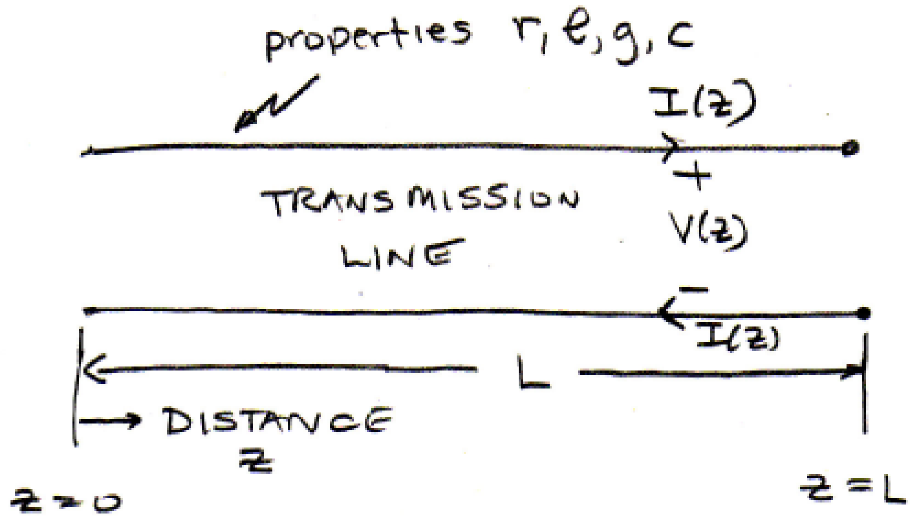
Previous mid-term tests with solutions are available from the course web site.

About the Mid-Term Test
The midterm test is a multiple-choice test:
•Each question has four numerical answers plus “none of these.”
•Circle the correct numerical answer on the exam paper.
•The test is closed book, so no textbook or notes are allowed.
•A sheet of formulas is attached to the test paper.
•You can get the formula sheet from the course web site.
•The mid-term test is “practice” for the final examination.
•The mid-term test will include a question on the “self-learning” topic of A.C. Circuit Analysis.

1

Transmission Line Circuits in the Sinusoidal Steady State

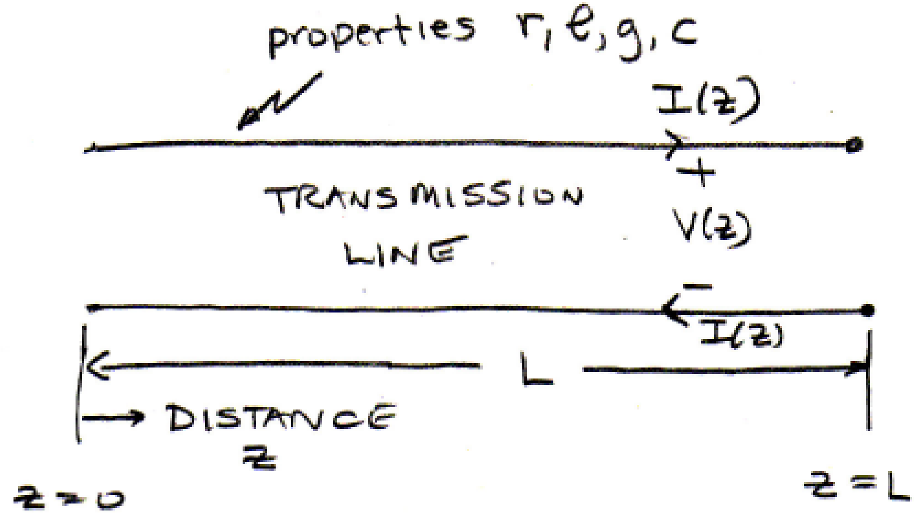
Inan and Inan Section 2.2

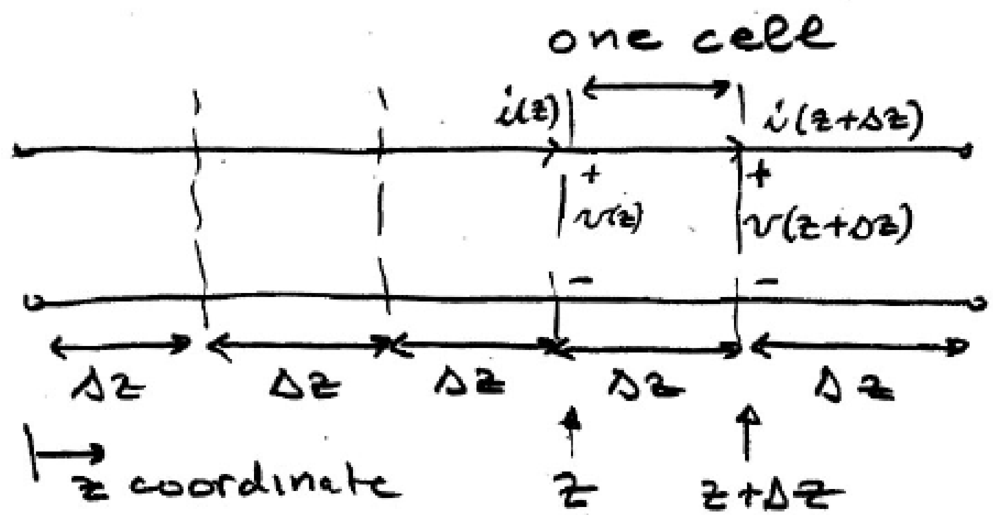


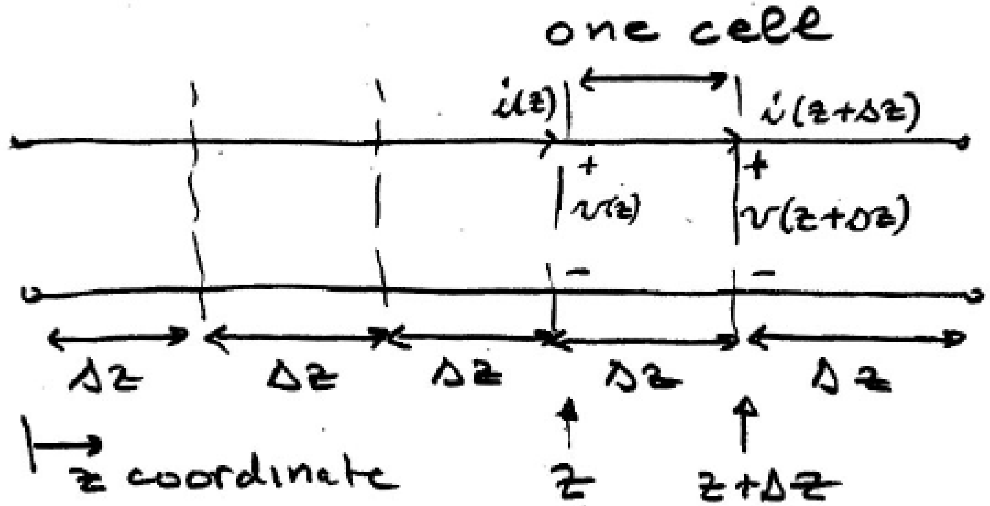
The transmission line parameters are:

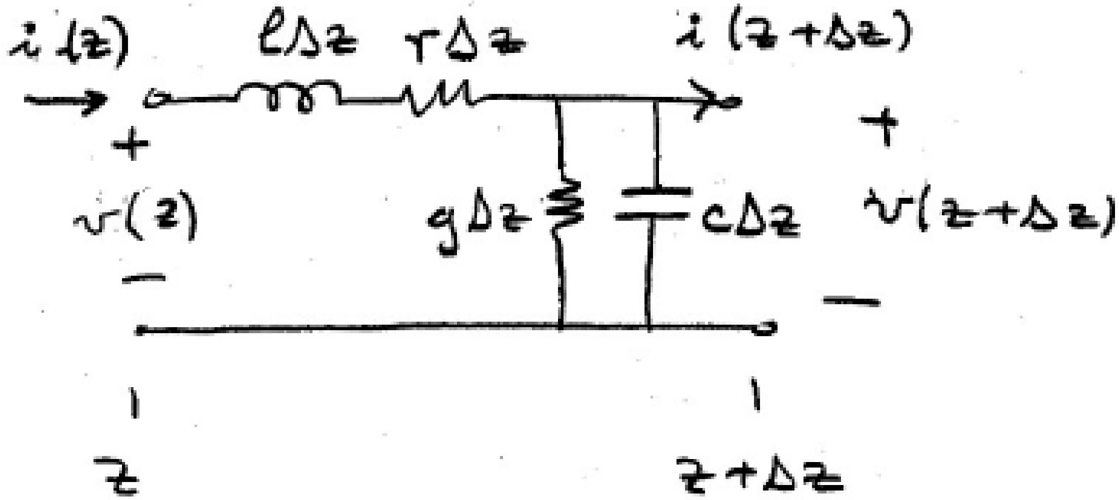
- ℓ H/m = inductance per unit length
- c F/m = capacitance per unit length
- r ohms/meter = series resistance per unit length
  - series resistance arises because of the currents flowing in the metal of the conductors
- g Siemens/meter = shunt conductance per unit length

Lossy Transmission Line Equations

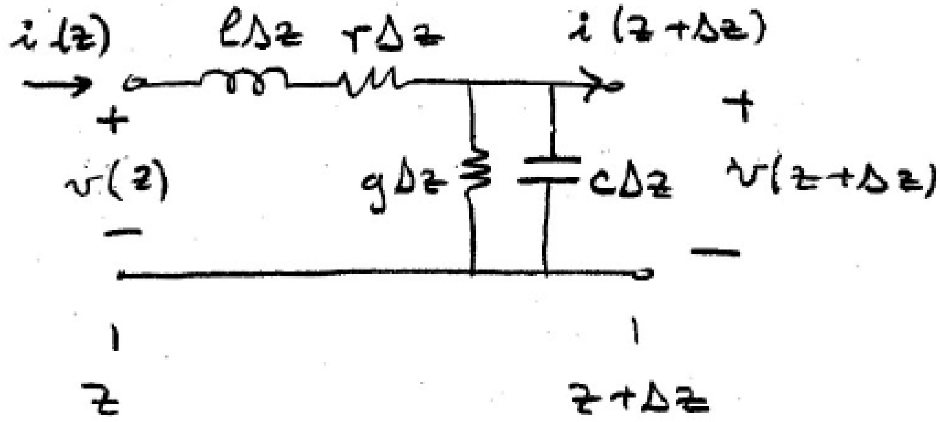





one cell
i(z)
i(z+Δz)
+
v(z)
−
+
v(z+Δz)
−
Δz
Δz
Δz
Δz
Δz
z coordinate
z
z+Δz


i(z)
lΔz
rΔz
i(z+Δz)
+
v(z)
−
gΔz
cΔz
+
v(z+Δz)
−
z
z+Δz

KVL Equation



KVL for the cell states:

$$v(z) - r\Delta z i - \ell \Delta z \frac{\partial i}{\partial t} - v(z+\Delta z) = 0$$

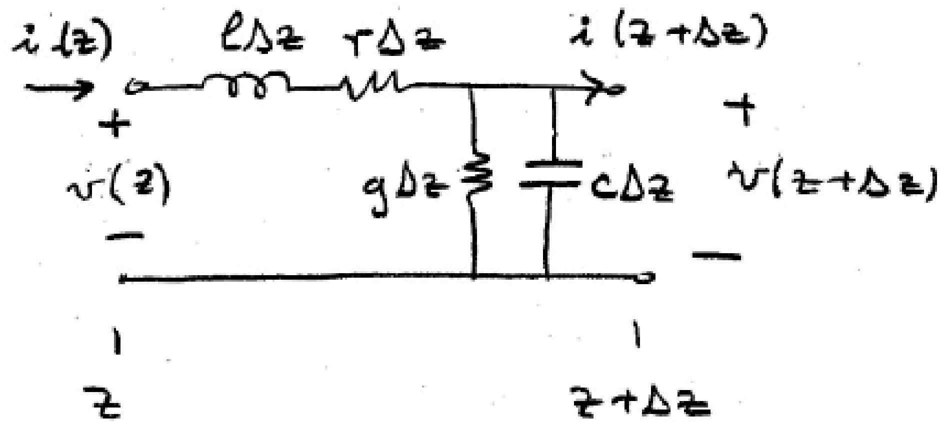
Rearrange the KVL equation:

$$v(z+\Delta z) - v(z) = -r\Delta z i - \ell \Delta z \frac{\partial i}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z) - v(z)}{\Delta z} = -ri - \ell \frac{\partial i}{\partial t}$$

$$\frac{\partial v}{\partial z} = -ri - \ell \frac{\partial i}{\partial t}$$

KCL Equation



KCL for the cell states:

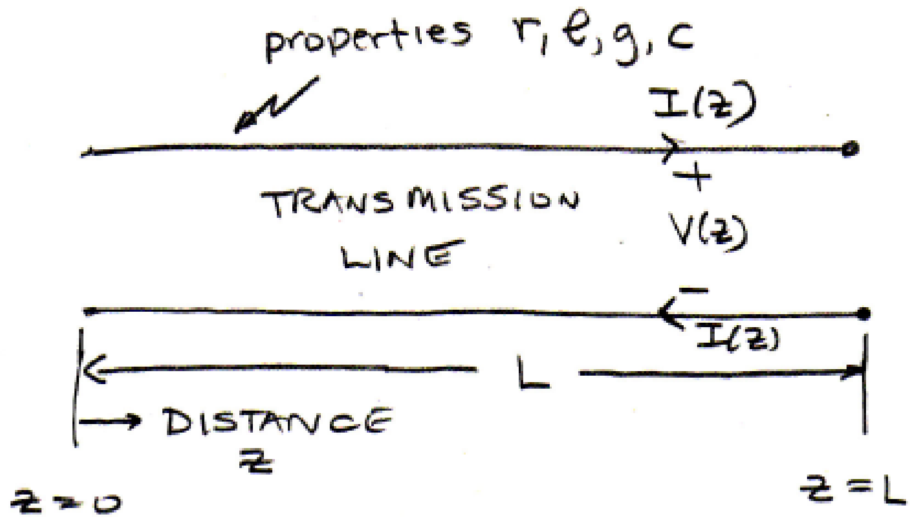
$$i(z) - g\Delta z v - c\Delta z \frac{\partial v}{\partial t} - i(z+\Delta z) = 0$$

Rearrange the KCL equation:

$$i(z+\Delta z) - i(z) = -g\Delta z v - c\Delta z \frac{\partial v}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z) - i(z)}{\Delta z} = -gv - c\frac{\partial v}{\partial t}$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t}$$



We have shown that:

$$\frac{\partial v}{\partial z} = -ri - \ell\frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t}$$

These equations are called “lossy” transmission line equations or “lossy” Telegrapher’s Equations.

Transmission-Line Equations for the Phasor Voltage and Current

$$v(z,t) = A(z)\cos(\omega t + \theta(z)) \longleftrightarrow V(z) = A(z)e^{j\theta(z)}$$

$$i(z,t) = B(z)\cos(\omega t + \phi(z)) \longleftrightarrow I(z) = B(z)e^{j\phi(z)}$$

time differentiation $\dfrac{\partial}{\partial t}$ is equivalent to multiplication by $j\omega$

Homework: Prove this!

$$\frac{\partial v}{\partial z} = -ri - \ell\frac{\partial i}{\partial t} \quad \Leftrightarrow \quad \frac{dV}{dz} = -(r + j\omega\ell)I$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t} \quad \Leftrightarrow \quad \frac{dI}{dz} = -(g + j\omega c)V$$

Wave Equation for Phasors

$$\frac{dV}{dz} = -(r + j\omega\ell)I$$

$$\frac{dI}{dz} = -(g + j\omega c)V$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\frac{dI}{dz}$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\left[-(g + j\omega c)V\right]$$

$$\frac{d^2V}{dz^2} = (r + j\omega\ell)(g + j\omega c)V$$

Define the "propagation constant" as $\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

Solution to the Wave Equation

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

Proof that $V = V^+ e^{-\gamma z}$ satisfies the wave equation:

- $\frac{dV}{dz} = \frac{d}{dz}(V^+ e^{-\gamma z}) = -\gamma V^+ e^{-\gamma z}$
- $\frac{d^2V}{dz^2} = \frac{d}{dz}(-\gamma V^+ e^{-\gamma z}) = (-\gamma)(-\gamma)V^+ e^{-\gamma z} = \gamma^2 V^+ e^{-\gamma z} = \gamma^2 V$
- Hence $\frac{d^2V}{dz^2} = \gamma^2 V$ and $V = V^+ e^{-\gamma z}$ does indeed satisfy the wave equation.
- Homework: prove that $V = V^- e^{\gamma z}$ satisfies the wave equation.

Propagation Constant

$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$$

$$\gamma = \alpha + j\beta$$

- α is the "attenuation constant" in "Nepers/meter"
  - "Neper" is a "unitless" unit like "radian".
- β is the "phase constant" in radians/meter.

Current on the Transmission Line

$$\frac{dV}{dz} = -(r + j\omega\ell)I$$

$$\frac{dI}{dz} = -(g + j\omega c)V$$

$$V(z) = V^{+}e^{-\gamma z} + V^{-}e^{\gamma z}$$

$$I(z) = \frac{-1}{(r + j\omega\ell)}\frac{dV}{dz}$$

$$I(z) = \frac{-1}{(r + j\omega\ell)}\frac{d}{dz}\left(V^{+}e^{-\gamma z} + V^{-}e^{\gamma z}\right)$$

$$I(z) = \frac{-1}{(r + j\omega\ell)}\left(-\gamma V^{+}e^{-\gamma z} + \gamma V^{-}e^{\gamma z}\right)$$

$$I(z) = \frac{\gamma}{(r + j\omega\ell)}\left(V^{+}e^{-\gamma z} - V^{-}e^{\gamma z}\right)$$

Characteristic Impedance

$$V = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{\gamma}{(r + j\omega\ell)}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

- The "characteristic resistance" R_c was defined earlier in the course as

  $R_c = \sqrt{\frac{\ell}{c}}$ ohms

- For a "lossy" transmission line, define the "characteristic impedance" Z_c as

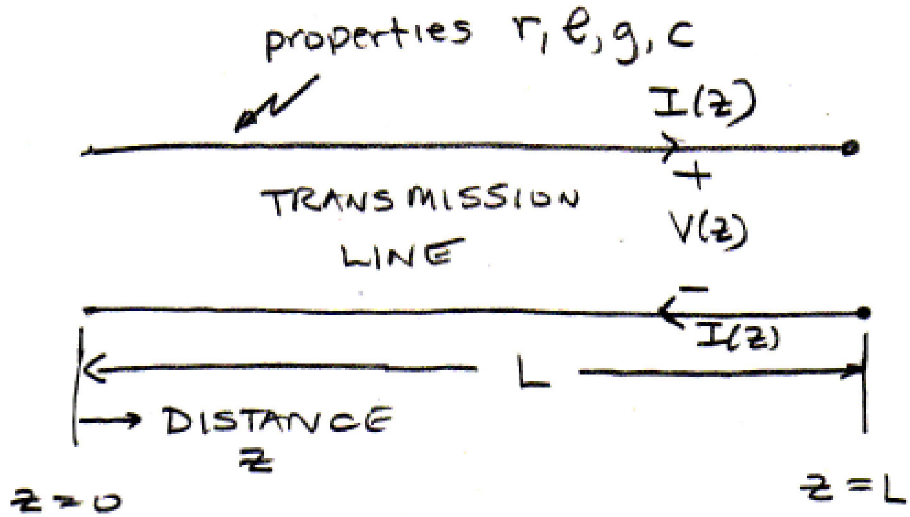
  $$Z_c = \frac{r + j\omega\ell}{\gamma} = \frac{r + j\omega\ell}{\sqrt{(r + j\omega\ell)(g + j\omega c)}} = \sqrt{\frac{(r + j\omega\ell)^2}{(r + j\omega\ell)(g + j\omega c)}}$$

  $Z_c = \sqrt{\frac{r + j\omega\ell}{g + j\omega c}}$ ohms

- Then the current on the transmission line is

  $$I(z) = \left(\frac{V^+}{Z_c} e^{-\gamma z} - \frac{V^-}{Z_c} e^{\gamma z}\right)$$

Summary-Sinusoidal Steady State



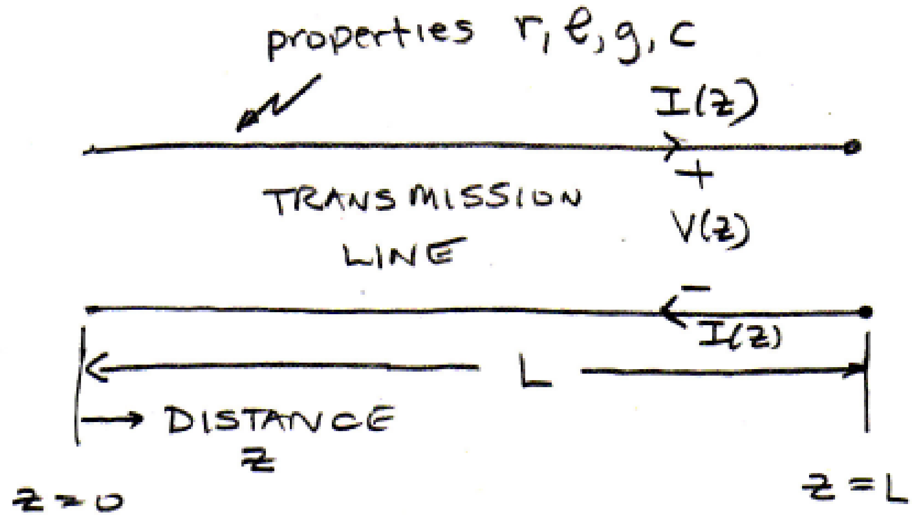
$$\frac{dV}{dz} = -(r + j\omega\ell)I \qquad \frac{d^2V}{dz^2} = \gamma^2 V$$

$$\frac{dI}{dz} = -(g + j\omega c)V \qquad \gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$$

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{V^+}{Z_c} e^{-\gamma z} - \frac{V^-}{Z_c} e^{\gamma z} \qquad Z_c = \sqrt{\frac{r + j\omega\ell}{g + j\omega c}}$$

Attenuation with Distance Travelled



$$V(z) = V^+ e^{-\gamma z}$$

$V^+ = |V^+| e^{j\theta}$ $\quad$ $V(z) = |V^+| e^{j\theta}\, e^{-\gamma z}$

$\gamma = \alpha + j\beta$ $\quad$ $V(z) = V^+ e^{-\gamma z} = |V^+| e^{j\theta}\, e^{-(\alpha + j\beta)z} = |V^+| e^{-\alpha z} e^{j(-\beta z + \theta)}$

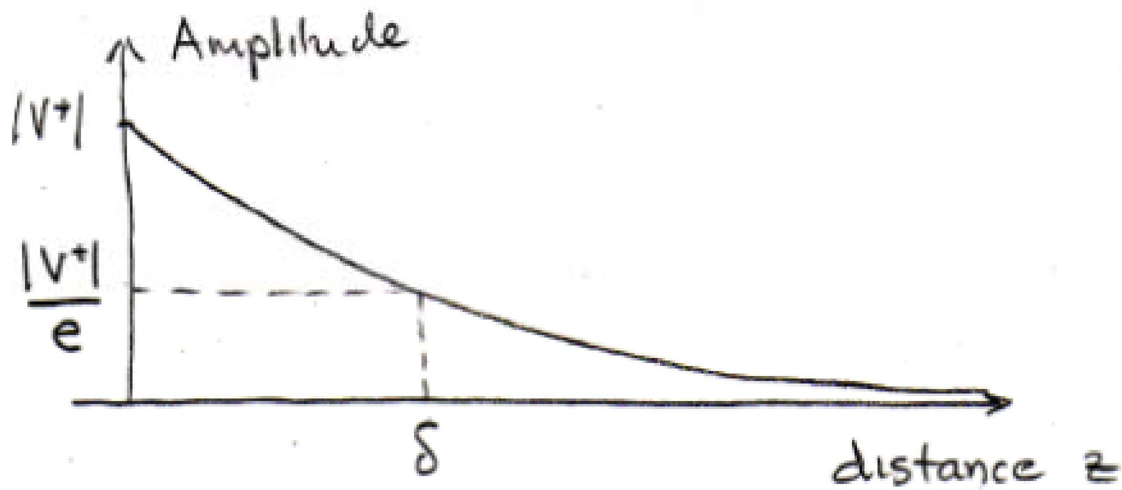
$$v(t) = |V^+| e^{-\alpha z} \cos(\omega t - \beta z + \theta)$$

The amplitude is $|V^+| e^{-\alpha z}$

Exponential Attenuation

$$v(t) = |V^+| e^{-\alpha z} \cos(\omega t - \beta z + \theta)$$

The amplitude of $v(t)$ is $|V^+| e^{-\alpha z}$

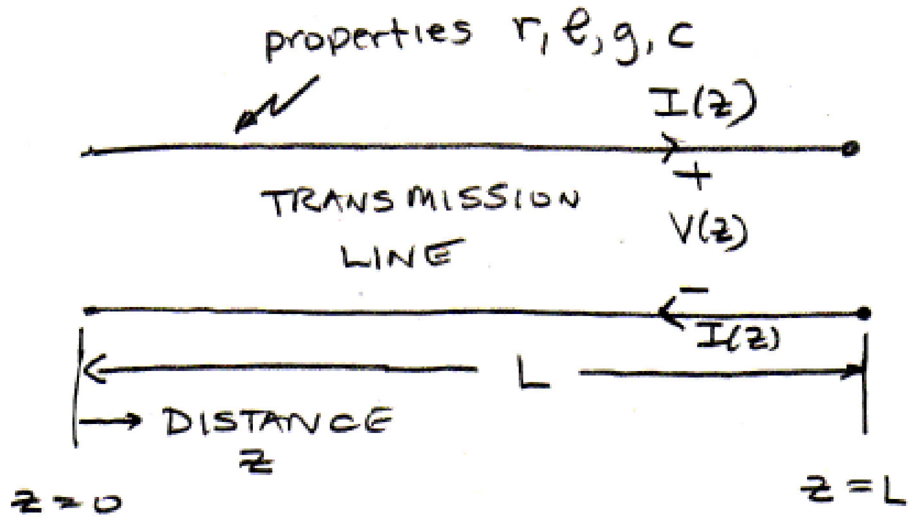


- The distance δ required for the amplitude to decrease by a factor of $1/e$ is called the "penetration depth", or for metals the "skin depth".

$$|V^+| e^{-\alpha\delta} = \frac{|V^+|}{e} \quad \text{so} \quad e^{-\alpha\delta} = \frac{1}{e} \quad \text{so} \quad \delta = \frac{1}{\alpha}$$

Lossless Transmission Lines

Inan and Inan Section 3.1



On a “lossless” transmission line, the series-resistance-per-unit-length is zero ($r = 0$) and the shunt-conductance-per-unit-length is zero ($g = 0$).

$$\frac{dV}{dz} = -(r + j\omega\ell)I \quad \Rightarrow \quad \frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -(g + j\omega c)V \quad \Rightarrow \quad \frac{dI}{dz} = -j\omega c V$$

Lossless Wave Equation

$$\frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -j\omega c V$$

$$\frac{d^2V}{dz^2} = -j\omega\ell\frac{dI}{dz} = -j\omega\ell(-j\omega cV) = -\omega^2\ell cV$$

$$\frac{d^2V}{dz^2} = -\omega^2\ell cV$$

$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)} = \sqrt{(j\omega\ell)(j\omega c)} = j\omega\sqrt{\ell c}$$

$\gamma = \alpha + j\beta$

So for "lossless" lines:

- $\alpha = 0$ so $\gamma = j\beta$
- $\beta = \omega\sqrt{\ell c}$

$\beta^2 = \left(\omega\sqrt{\ell c}\right)^2 = \omega^2\ell c$

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

Solution to the Lossless Wave Equation

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

(Homework: prove this is a solution to $\frac{d^2V}{dz^2} = -\beta^2 V$ by direct substitution.)

In the time domain:

$$V^+ = |V^+| e^{j\theta^+} = C^+ e^{j\theta^+}$$

$$V^- = |V^-| e^{j\theta^-} = C^- e^{j\theta^-}$$

$$V(z) = C^+ e^{j\theta^+} e^{-j\beta z} + C^- e^{j\theta^-} e^{j\beta z}$$

$$V(z) = C^+ e^{j(-\beta z + \theta^+)} + C^- e^{j(\beta z + \theta^-)}$$

$$v(z,t) = C^+ \cos(\omega t - \beta z + \theta^+) + C^- \cos(\omega t + \beta z + \theta^-)$$

Find the Current on the Lossless Transmission Line

$$\frac{dV}{dz} = -j\omega \ell I \qquad \beta = \omega\sqrt{\ell c}$$

$$\frac{dI}{dz} = -j\omega c V \qquad V(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}$$

$$I = \frac{-1}{j\omega\ell}\frac{dV}{dz} = \frac{-1}{j\omega\ell}\frac{d}{dz}\left(V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}\right)$$

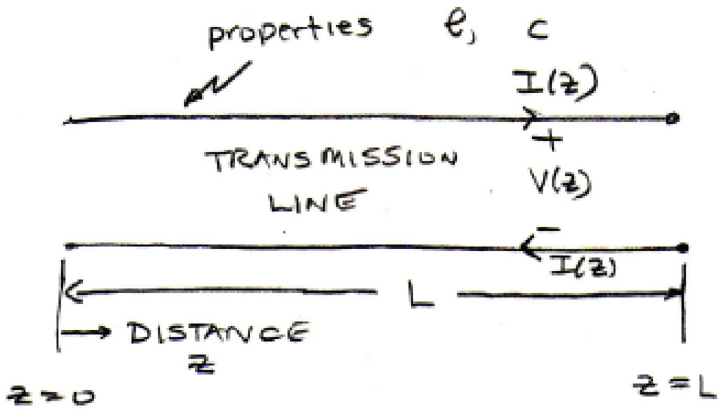
$$I = \frac{-1}{j\omega\ell}\frac{dV}{dz} = \left(\frac{-1}{j\omega\ell}\right)\left(-j\beta V^{+}e^{-j\beta z} + j\beta V^{-}e^{j\beta z}\right)$$

$$I = \frac{\beta}{\omega\ell}V^{+}e^{-j\beta z} - \frac{\beta}{\omega\ell}V^{-}e^{j\beta z}$$

$$R_c = \frac{\omega\ell}{\beta} = \frac{\omega\ell}{\omega\sqrt{\ell c}} = \sqrt{\frac{\ell}{c}} \text{ ohms}$$

$$I(z) = \frac{V^{+}}{R_c}e^{-j\beta z} - \frac{V^{-}}{R_c}e^{j\beta z}$$

Summary – Lossless Transmission Lines



Lossless transmission-line equations:

$$\frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -j\omega c V$$

Lossless wave equation:

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

where $\beta = \omega\sqrt{\ell c}$ is the phase constant.

Characteristic Resistance

$$R_c = \sqrt{\frac{\ell}{c}} \text{ ohms}$$

Voltage and Current

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

Find the Wavelength

$$V = Ae^{-j\beta z}$$

$$v(z,t) = \text{Re}[Ve^{j\omega t}] = \text{Re}[Ae^{-j\beta z}e^{j\omega t}] = \text{Re}[Ae^{j(\omega t-\beta z)}]$$
$$v(z,t) = \text{Re}[A\cos(\omega t-\beta z) + jA\sin(\omega t-\beta z)]$$
$$v(z,t) = A\cos(\omega t-\beta z)$$

At time t_1, plot $v(z,t_1) = A\cos(\omega t_1 - \beta z)$ as a function of distance z.

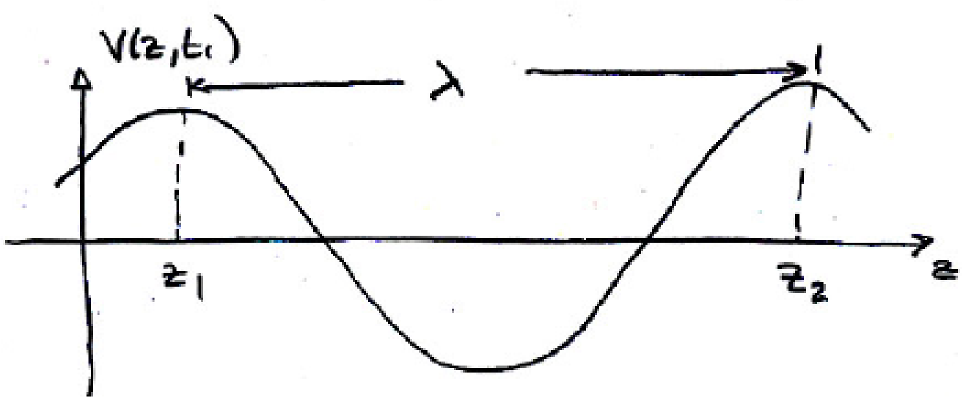
The voltage has a maximum when $(\omega t_1 - \beta z) = 0, \pm 2\pi, \pm 4\pi, \ldots$

- There is a maximum when $\omega t_1 - \beta z_1 = 0$ so

$$z_1 = \frac{\omega t_1}{\beta}$$

- And there is another maximum when $\omega t_1 - \beta z_2 = -2\pi$ so

$$z_2 = \frac{\omega t_1 + 2\pi}{\beta} = \frac{\omega t_1}{\beta} + \frac{2\pi}{\beta} = z_1 + \frac{2\pi}{\beta}$$



The wavelength is the distance between the adjacent maxima:

$$\lambda = z_2 - z_1 = \frac{\omega t_1 + 2\pi}{\beta} - \frac{\omega t_1}{\beta} = \frac{2\pi}{\beta}$$

Find the Speed of Travel

$$v(z,t) = A\cos(\omega t - \beta z)$$

How far does the wave travel between $t = t_1$ and $t = t_2$?

At time t_1, plot $v(z,t_1) = A\cos(\omega t_1 - \beta z)$ as a function of distance z.

The voltage has a maximum when $(\omega t_1 - \beta z) = 0, \pm 2\pi, \pm 4\pi, \ldots$

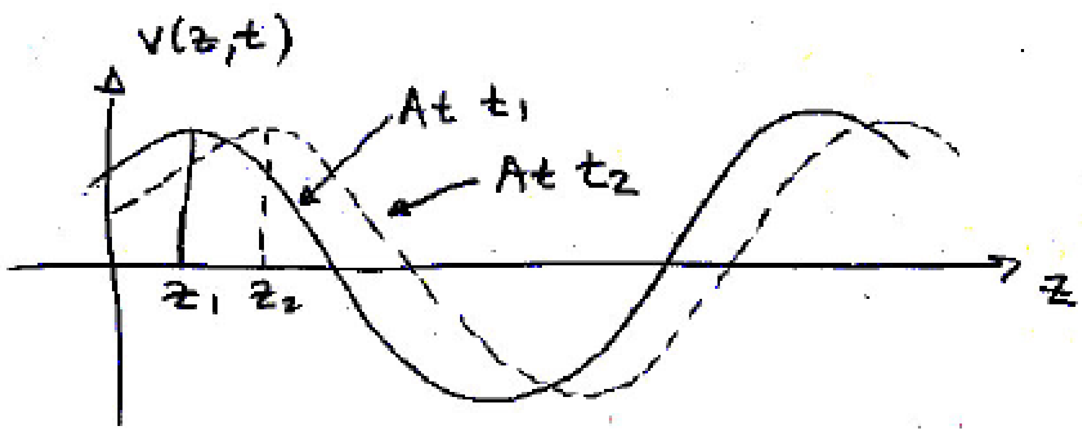
- There is a maximum when $\omega t_1 - \beta z_1 = 0$ so

$$z_1 = \frac{\omega t_1}{\beta}$$

Now plot $v(z,t_2) = A\cos(\omega t_2 - \beta z)$ at some later time $t_2 > t_1$.

Voltage $v(z,t_2)$ has a maximum when $\omega t_2 - \beta z_2 = 0$ or $z_2 = \frac{\omega t_2}{\beta}$

Since $t_2 > t_1$, we have $z_2 > z_1$



the speed of travel or "phase velocity" is

$$u = \frac{z_2 - z_1}{t_2 - t_1} = \frac{\frac{\omega t_2}{\beta} - \frac{\omega t_1}{\beta}}{t_2 - t_1} = \frac{\omega}{\beta}$$

Wavelength and Speed of Travel

In general the wavelength is $\lambda = \dfrac{2\pi}{\beta}$

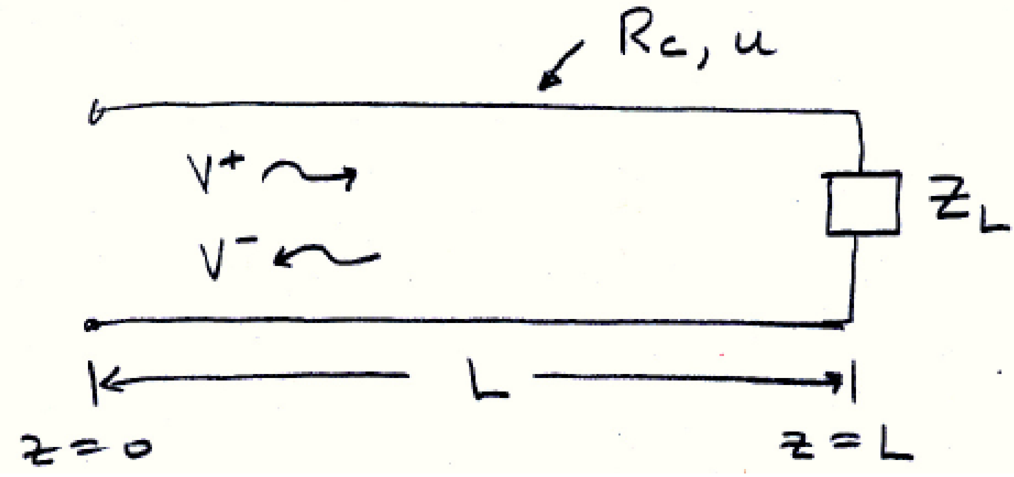
In general the speed of travel is $u = \dfrac{\omega}{\beta}$

For a lossless transmission line $\beta = \omega\sqrt{\ell c}$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\ell c}} = \frac{1}{\sqrt{\ell c}} \qquad u = \frac{1}{\sqrt{\ell c}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega/u} = \frac{2\pi u}{2\pi f} = \frac{u}{f} \qquad \lambda = \frac{u}{f}$$

Transmission Line Terminated with a Load



Inan and Inan Sections 3.2 and 3.3

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

If we know V^+, can we find V^- ?

At the load, we must satisfy $V(L) = Z_L I(L)$

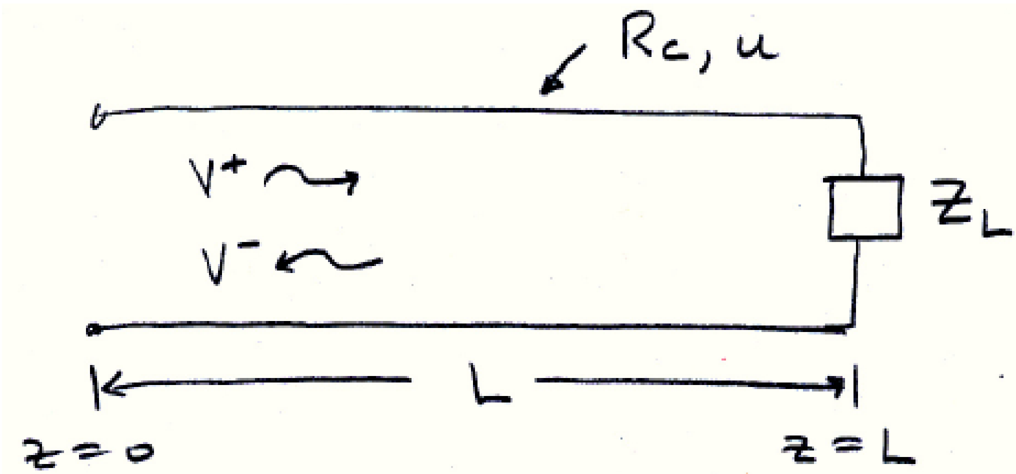
$$V(L) = V^+ e^{-j\beta L} + V^- e^{j\beta L}$$

$$I(L) = \frac{V^+}{R_c} e^{-j\beta L} - \frac{V^-}{R_c} e^{j\beta L}$$

$$V^+ e^{-j\beta L} + V^- e^{j\beta L} = Z_L \frac{V^+}{R_c} e^{-j\beta L} - Z_L \frac{V^-}{R_c} e^{j\beta L}$$

$$V^- = \frac{Z_L - R_c}{Z_L + R_c} e^{-j2\beta L} V^+$$

Reflection Coefficient at the Load



$$\Gamma_L = \frac{V^- e^{j\beta L}}{V^+ e^{-j\beta L}}$$

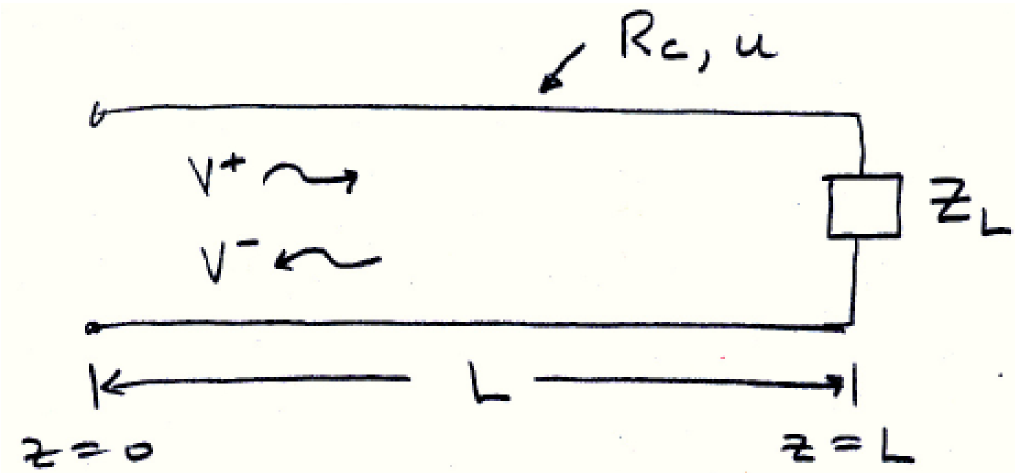
$$V^- = \frac{Z_L - R_c}{Z_L + R_c} e^{-2j\beta L} V^+$$

$$\Gamma_L = \frac{\frac{Z_L - R_c}{Z_L + R_c} e^{-2j\beta L} V^+ e^{j\beta L}}{V^+ e^{-j\beta L}}$$

$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c}$$

$$V^- = \Gamma_L e^{-j2\beta L} V^+$$

Matched Load



$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c}$$

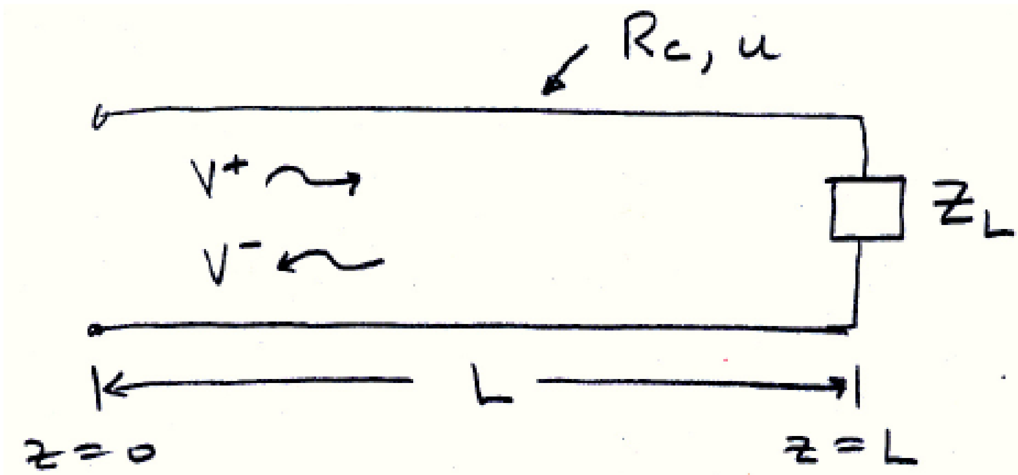
Matched Load $Z_L = R_c$:

- The reflection coefficient is

$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c} = \frac{R_c - R_c}{R_c + R_c} = 0$$

- The reflected voltage is $V^- = \Gamma_L e^{-j2\beta L} V^+ = 0$

Voltage and Current



Voltage and Current

$$V(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}$$

$$I(z) = \frac{V^{+}}{R_c}e^{-j\beta z} - \frac{V^{-}}{R_c}e^{j\beta z}$$

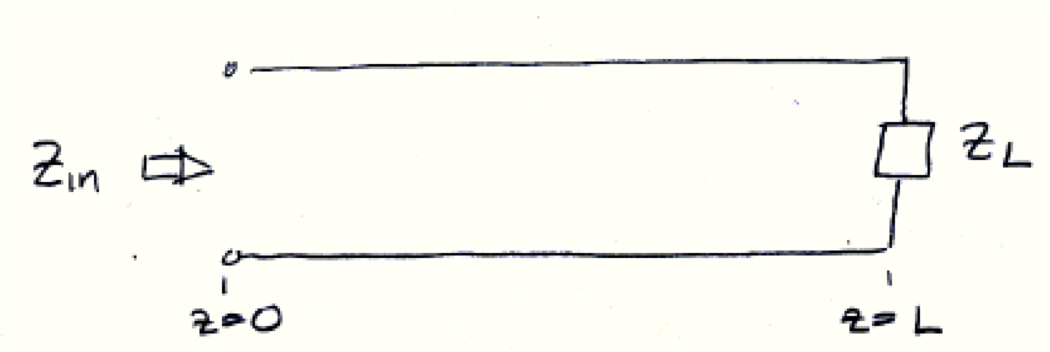
$$V^{-} = \Gamma_L e^{-j2\beta L}V^{+}$$

$$\begin{aligned} V(z) &= V^{+}e^{-j\beta z} + V^{-}e^{j\beta z} \\ &= V^{+}e^{-j\beta z} + \Gamma_L e^{-j2\beta L}V^{+}e^{j\beta z} \\ &= V^{+}\left(e^{-j\beta z} + \Gamma_L e^{-j2\beta L}e^{j\beta z}\right) \end{aligned}$$

$$V(z) = V^{+}e^{-j\beta L}\left(e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)}\right)$$

$$I(z) = V^{+}e^{-j\beta L}\left(\frac{1}{R_c}e^{-j\beta(z-L)} - \frac{1}{R_c}\Gamma_L e^{j\beta(z-L)}\right)$$

The Input Impedance



$$V(z)=V^+e^{-j\beta L}\left(e^{-j\beta(z-L)}+\Gamma_L e^{j\beta(z-L)}\right)$$

$$I(z)=V^+e^{-j\beta L}\left(\frac{1}{R_c}e^{-j\beta(z-L)}-\frac{1}{R_c}\Gamma_L e^{j\beta(z-L)}\right)$$

$$\Gamma_L=\frac{Z_L-R_c}{Z_L+R_c}$$

$$Z_{in}=\frac{V(0)}{I(0)}$$

$$V(0)=V^+e^{-j\beta L}\left(e^{-j\beta(0-L)}+\Gamma_L e^{j\beta(0-L)}\right)$$

$$V(0)=V^+e^{-j\beta L}\left(e^{j\beta L}+\Gamma_L e^{-j\beta L}\right)$$

$$I(0)=V^+e^{-j\beta L}\left(\frac{1}{R_c}e^{-j\beta(0-L)}-\frac{1}{R_c}\Gamma_L e^{j\beta(0-L)}\right)$$

$$I(0)=V^+e^{-j\beta L}\left(\frac{1}{R_c}e^{j\beta L}-\frac{1}{R_c}\Gamma_L e^{-j\beta L}\right)$$

$$Z_{in}=\frac{V(z=0)}{I(z=0)}=\frac{V^+e^{-j\beta L}\left(e^{j\beta L}+\Gamma_L e^{-j\beta L}\right)}{V^+e^{-j\beta L}\left(\frac{1}{R_c}e^{j\beta L}-\frac{1}{R_c}\Gamma_L e^{-j\beta L}\right)}$$

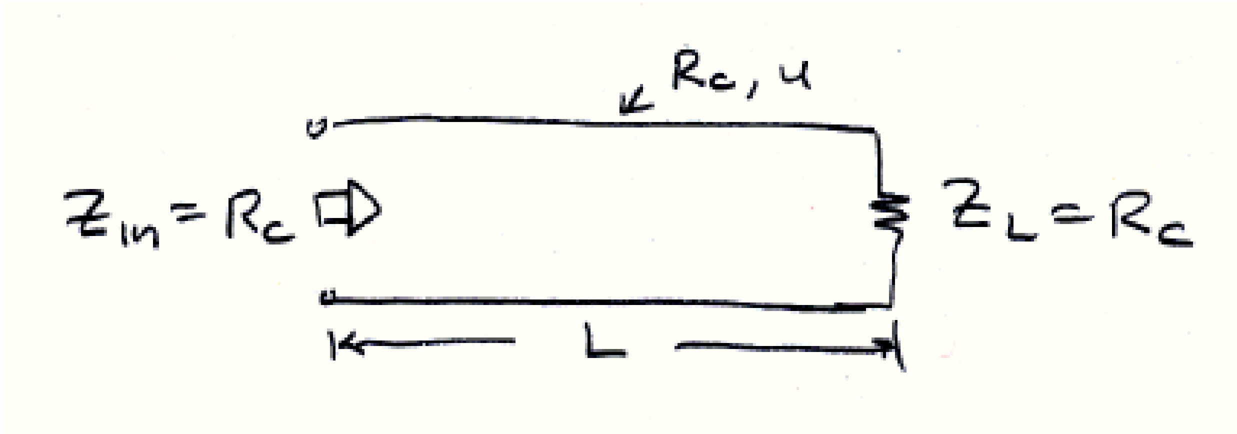
$$Z_{in}=R_c\frac{Z_L+jR_c\tan\beta L}{R_c+jZ_L\tan\beta L}$$

Homework: do the missing algebra to derive this formula.

Important Special Cases

Matched Load

- What is the input impedance of a transmission line terminated with a matched load?



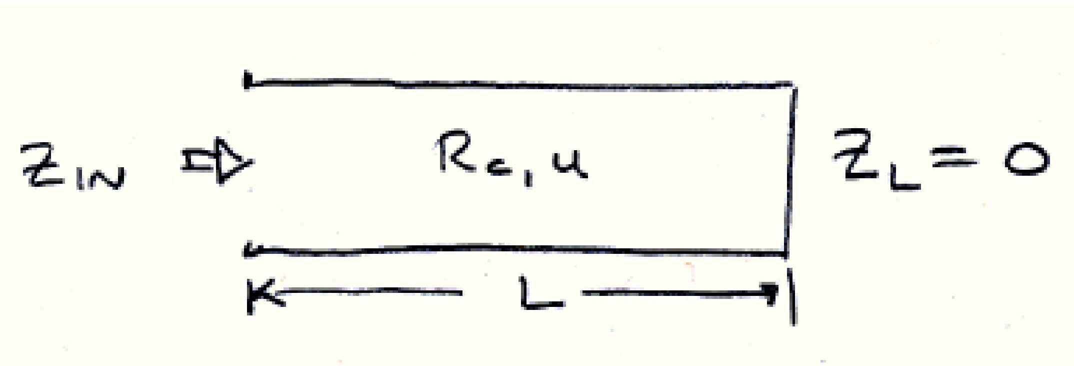
- If $Z_L = R_c$, then the input impedance is

$$Z_{in} = R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L} = R_c \frac{R_c + jR_c \tan \beta L}{R_c + jR_c \tan \beta L} = R_c$$

- So the input impedance of a transmission line terminated with a matched load is $Z_{in} = R_c$.

Short-Circuited Transmission Line

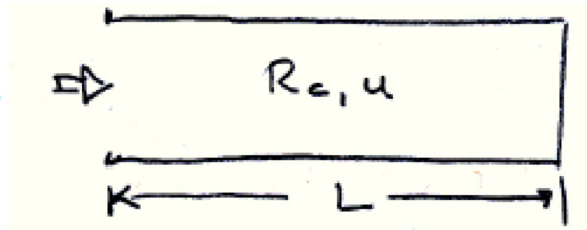
- What is the input impedance of a transmission line terminated with a short circuit?



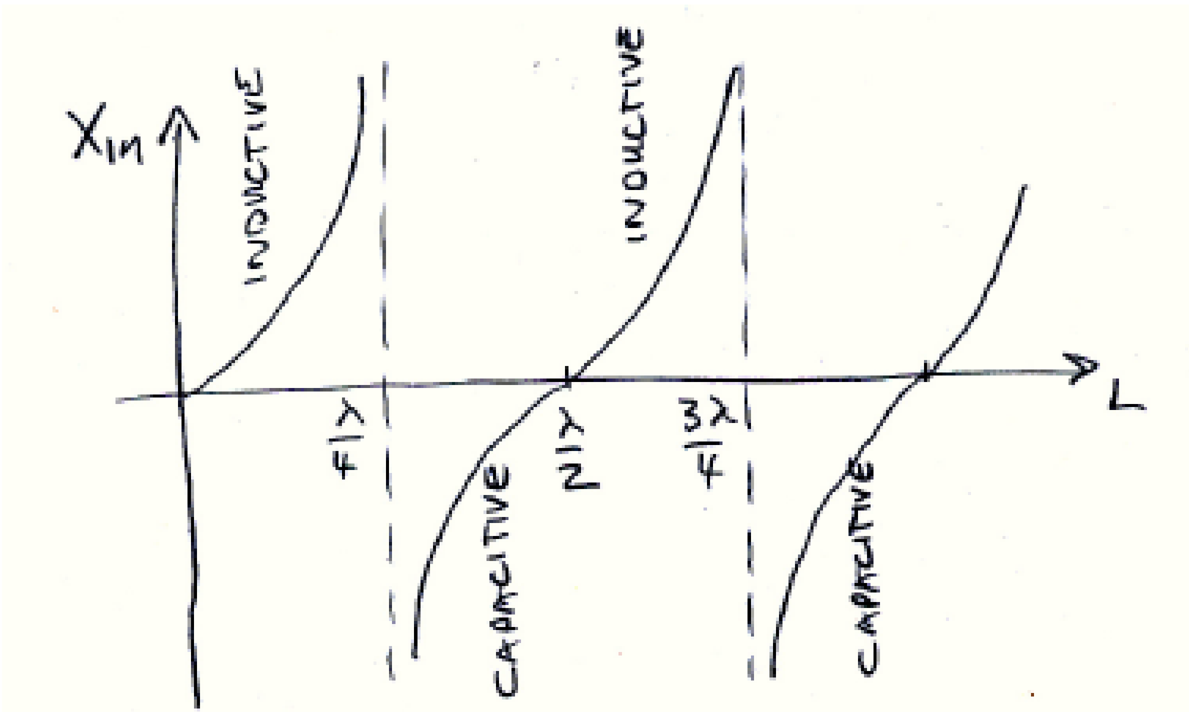
- A short length of transmission line terminated with a short circuit is called a "stub" and can be used for impedance matching. (We will do this later in the course.)
- What is the input impedance of a transmission line terminated with a short circuit, $Z_L = 0$?

$$Z_{in} = R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L} = R_c \frac{0 + jR_c \tan \beta L}{R_c + j0 \tan \beta L} = jR_c \tan \beta L$$

$$Z_{in} = jR_c \tan \beta L$$



- $Z_{in} = jR_c \tan \beta L$ is "reactive", meaning that it is pure imaginary.
- We can write $Z_{in} = jX_{in}$ where the reactance is $X_{in} = R_c \tan \beta L$



When $\beta L \to \frac{\pi}{2}$, $\tan \beta L \to +\infty$.

$$L = \frac{\pi}{2\beta} = \frac{\pi}{2\frac{2\pi}{\lambda}} = \frac{\lambda}{4}$$

Open-Circuited Transmission Line

- What is the input impedance of a transmission line terminated with an open circuit, $Z_L \to \infty$?

$$Z_{in} = \lim_{Z_L \to \infty} R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L}$$

$$Z_{in} = R_c \frac{Z_L}{jZ_L \tan \beta L} = -jR_c \cot \beta L$$

- $Z_{in} = -jR_c \cot \beta L$ is "reactive" meaning pure imaginary.
- We can write $Z_{in} = jX_{in}$ where the reactance is $X_{in} = -R_c \cot \beta L$