Quad-Rotor UAV: High-Fidelity Modeling and Nonlinear PID Control

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Quad-rotor helicopter is an Unmanned Aerial Vehicle (UAV), whose lift is generated by four rotors located on the corner of X-shape. Due to simplicity of its dynamics and its ability to hover, quad-rotor helicopter becomes as a popular platform for UAV. Current designs mostly consider a linear model for controller design. In this paper, we derive nonlinear dynamic equations of the quad-rotor UAV for the hovering motion from basic Newton’s second law. Nonlinear Proportional Integral Derivative (PID) controller is proposed to control the quad-rotor UAV. This controller uses the speed and the orientation of each propeller in hovering motion of the quad-rotor UAV. A number of trajectories are used to demonstrate the effectiveness of the designed controller.

Nomenclature

\(M\) = mass of the vehicle
\(x, y, z\) = \(x, y,\) and \(z\) position of the vehicle, respectively
\(\phi, \theta, \psi\) = roll, pitch, and yaw angle of the vehicle, respectively
\(I_{xx}, I_{yy}, I_{zz}\) = mass moment of inertia about \(x, y,\) and \(z\) axis, respectively
\(g\) = acceleration of the gravity
\(F_g\) = gravitational force in the earth frame
\(F_{gb}\) = gravitational force in the vehicle frame
\(C_{d(x,y,z)}\) = drag coefficients in the \(x, y,\) and \(z\) direction, respectively
\(F_d(x,y,z)\) = drag forces in the \(x, y,\) and \(z\) direction, respectively
\(F_T(x,y,z)\) = thrust forces in the \(x, y,\) and \(z\) direction, respectively
\(R_{(1,2,3,4)}\) = the orientation angles of each propeller
\((\bar{x}, \bar{y})\) = current position of the vehicle
\((\bar{x}, \bar{y})\) = command position of the vehicle
\(\Pi\) = transformation matrix
\(v_{(x,y,z)}\) = velocity in the \(x, y,\) and \(z\) direction, respectively
\(L, M, N\) = the moments produced by the thrust force around \(x, y,\) and \(z\) axis, respectively

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I. Introduction

Unmanned Aerial Vehicles (UAVs) are remotely operated or autonomous piloted vehicles. These vehicles have been used in applications such as security, management of natural risks, for intervention in hostile environments, management of ground installations, agriculture, and military\textsuperscript{1,3}. Use of UAVs makes it possible to gather information in dangerous environments without risk to flight crews.

UAVs are classified into two categories: fixed- and rotary-wing types. The rotary-wing type UAVs are more advantageous than the fixed-wing type UAVs in the sense of Vertical Take-Off and Landing (VTOL) capability, omni-directional flying, and hovering performance, and can be divided into quad-rotor and other helicopter types based on their shapes\textsuperscript{4-5}. The advantages of the quad-rotor versus comparably scale helicopters are that quad-rotor does not require mechanical linkages to vary rotor angle of attack as they spin, the use of four rotors allows each individual rotor to have a smaller diameter than the equivalent helicopter rotor allowing them to store less kinetic energy during flight and this reduces the damage caused by the rotors hitting any objects, and by enclosing the rotors within a frame, the rotors can be protected during collisions.

Depending on the flying principal, we can classify the aerial vehicles in multiple categories depending on the flying principal as shown in Fig. 1\textsuperscript{5}. In the motorized class, the bird like Micro Aerial Vehicle (MAV) can be considered as the perfect example for the fast navigation. VTOL is classified within the same category. The most famous example of the quad-rotor UAV is the Draganflyer which is shown in Fig. 2. This UAV is a radio-controlled four-rotor helicopter manufactured by Draganflyer\textsuperscript{6}. Although it can fly simply, stabilization of the hovering is a challenge\textsuperscript{2}. Control of the quad-rotor has been widely studied in a number of studies\textsuperscript{7-11}. The quad-rotor helicopter of the Stanford Testbed of Autonomous Rotorcraft for Multi-Agent Control II (STARMAC II) has been caused by moments that affect altitude and control, and thrust variation that affects altitude control. The results of this work have proven the insufficiency for accurate trajectory tracking of the vehicle at speed and in uncontrolled environments but the control design could be improved by careful consideration of these disturbances\textsuperscript{12}. Omni-directional Stationary Flying Outstretched Robot OS4 is a quad-rotor project developed towards fully autonomous operation. The main objective of this flying robot is the development and implementation of an active control system for this vehicle\textsuperscript{5,13}.

Although many control algorithms have obtained good results in the previous works, those works focused primarily on the control of the vehicle by taking into account only the speed of the propeller. The objective of this work is to present the dynamic model of the quad-rotor and propose a nonlinear Proportional-Integral-Derivative (PID) controller based on the adjustment of the orientation of the propeller in addition to control the speed of the rotor. The main contribution of our work is to design PID controller for quad-rotor vehicle based on the orientation and the speed of each propeller. In this paper, we analyze the high-fidelity model of the quad-rotor
and obtain the equations of motion in Section II. In Section III, nonlinear PID controller is used to control the UAV model. In this section there are two subsections: the first one shows the control of the orientation of each propeller and the second one shows the control of the speed of the propellers. In Section VI, simulation results are presented and supported by figures. Finally, conclusions are presented.

II. Motion of the Quad-Rotor UAV

Quad-rotor is an aircraft without pilot whose lift is generated by its four rotors. It contains four propellers and each two of them forms a pair of propellers, the first pair is located on the x-axis and rotated clockwise and the second pair is located on the y-axis and rotated counterclockwise as shown in Fig. 3. Generally, its motion is controlled by varying the lift force produced by the rotors\(^{14}\). Each rotor produces both a thrust and a torque about its center of rotation\(^2\). The gyroscopic effects and the aerodynamic torques tend to cancel and these four rotors do not have a swash plate\(^{15}\).

The equations are presented about the body frame \((X, Y, Z)\). For any point of the airframe expressed in the earth-fixed frame \((X_E, Y_E, Z_E)\), one can write transformation matrix as:

\[
\Pi = \begin{bmatrix}
c \theta c \psi & -c \phi c \psi + s \phi s \theta \psi & s \phi s \psi + c \phi c \psi \\
-c \theta c \psi & c \phi c \psi + s \phi s \theta \psi & -s \phi c \psi + c \phi c \psi \\
-s \theta & s \phi c \theta & c \phi c \theta
\end{bmatrix}
\]

(1)

where \(s (\phi, \theta, \psi) = \sin (\phi, \theta, \psi)\), \(c (\phi, \theta, \psi) = \cos (\phi, \theta, \psi)\).

The equation of motion of the quad-rotor at inertial coordinate system is given as\(^4\):

\[
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} = \Pi \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}
\]

(2)

\(F_x, F_y, F_z\) are the net forces in an inertial system represented in the body coordinate system and \((x, y, z)\) represent the location of the vehicle in the earth-fixed frame and \(\Pi\) is defined as Eq. (1). It can be seen that the accelerations in the earth-fixed frame are obtained by multiplying the forces by the transformation matrix.

The main forces acting on the body are the propeller thrusts, drag forces and gravitational forces. The direction of the propeller thrust is perpendicular to the surface of rotating and its value is proportional with the speed of rotation\(^7\).

The propeller is driven by a DC motor mounted on the stepper motor located at the end of a crossing body frame as shown in Fig. 4. The frame \((x, y, z)\) of the DC motor has been assumed. The DC motor frame \((x, y, z)\) is parallel to the vehicle body frame \((X, Y, Z)\) when there is no orientation of the propeller, as demonstrated in the
classical quad-rotor. The orientation of the propeller will take effect and the \( y-z \) plane of the propeller will orient at an angle as the propeller is tilted as a result of the rotation of the stepper motor. So the thrust forces will no longer be perpendicular on the X-Y plane of the vehicle. To find the projection of the forces on each axis, we need to multiply the thrust force by the \( \text{sine} \) or \( \text{cosine} \) of the angle that the propeller is tilted. Figure 5 shows the orientation of propeller 1 and the components of the thrust force.

The drag force is proportional with the square of the velocity on each axis by coefficient of drag as shown in following equation.

\[
\begin{bmatrix}
    F_{dx} \\
    F_{dy} \\
    F_{dz}
\end{bmatrix} = \frac{1}{\Pi} \begin{bmatrix}
    C_{dx} \cdot v_x^2 \\
    C_{dy} \cdot v_y^2 \\
    C_{dz} \cdot v_z^2
\end{bmatrix}
\]  

(3)

Since the velocity of each axis is presented in earth frame, we need to divide it by the transformation matrix (1) to get its component on the body frame.

The main force acting on the quad-rotor is the gravity force\(^1\) equaling to the total mass of the quad-rotor multiplied by the acceleration of the gravity as following:

\[
F_g = M \cdot g
\]  

(4)

Its direction is toward the ground (negative z-direction of the earth frame). By dividing the gravitational force by the transformation matrix, we can calculate the effect on each axis on the body frame.

\[
F_{gb} = \begin{bmatrix}
- \sin \theta \cdot F_g \\
\cos \theta \cdot \sin \phi \cdot F_g \\
\cos \theta \cdot \cos \phi \cdot F_g
\end{bmatrix} = \begin{bmatrix}
F_{gb 1} \\
F_{gb 2} \\
F_{gb 3}
\end{bmatrix}
\]  

(5)

The equations of the forces affected to the quad-rotor are:

\[
F_x = F_{dx} + F_{T_x} + F_{gb 1}
\]  

(6)

\[
F_y = F_{dy} + F_{T_y} + F_{gb 2}
\]  

(7)

\[
F_z = F_{dz} + F_{T_z} + F_{gb 3}
\]  

(8)

By the derivatives of the acceleration, we can get the velocity and then the location of the vehicle:

\[
\ddot{\xi} = \int \dot{\xi} \, d\tau \quad \xi = \int \dot{\xi} \, d\tau \quad \text{where} \quad \xi = \begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
\]

\( \dot{\xi}_0 \) and \( \ddot{\xi}_0 \) are the initial conditions of the velocity and the position, respectively.

The roll equations of motion are:

\[
\begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix} \begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} = \begin{bmatrix}
L \\
M \\
N
\end{bmatrix}
\]  

(9)

\[\text{Figure 5. Propeller 1 with certain orientation. This figure shows the propeller 1 at an angle (R1). The thrust force should be multiplied by \( \cos(R1) \) to get the component of the force on the Z-axis of the vehicle body. This view is taken towards the positive X-axis.}\]
The main moments acting on the quad-rotor vehicle are produced by propellers, which is equal to the thrust force multiplied by the momentum arm. Note that the drag force was neglected in computing the moment. This force was found to cause a negligible disturbance on the total moment over the flight regime of interest.

\[ \dot{\eta} = \int_{\eta_0}^t \dot{\eta} \, d\tau, \quad \eta = \int_{\eta_0}^t \dot{\eta} \, d\tau, \quad \text{where} \quad \eta = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \]

\( \dot{\eta}_0 \) and \( \eta_0 \) are the initial conditions of the angular velocity and the Euler angles rotational components, respectively.

III. Controller Design

Nonlinear (PID) controller is proposed to design trajectory tracking controller for the quad-rotor UAV and it is based on two fundamentals: control of orientation of each propeller and control of speed of each motor. The parameters of the PID controller are tuning by trial and error.

a) Control the orientation of each propeller

We use the orientation primarily to change the yaw angle, which is the angle between a quad-rotor’s heading and a reference heading, to get the commanded angle. This is the yaw controller. The criterion of employing yaw control depends on the distance, which is denoted as \( d \), between the current position \((\bar{x}, \bar{y})\) and commanded position \((\bar{x}', \bar{y}')\), where the distance \( d \) is expressed as the following equation:

\[ d = \sqrt{x_e^2 + y_e^2} \] (10)

where \( x_e \) and \( y_e \) are defined as:

\[ x_e = \bar{x} - \tilde{x} \]
\[ y_e = \bar{y} - \tilde{y} \]

Then, following yaw control will be activated when this distance is larger than a specific value \( \gamma \), \( d > \gamma \), where \( \gamma \) is a positive constant. Assume \( \gamma = 3 \).

\[ \tilde{x} = \{ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_n \} \]
\[ \tilde{y} = \{ \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \ldots, \tilde{y}_n \} \]

where \( n \) is the number of the command points.

Then assuming the distance between the command points is

\[ \lambda = \sqrt{(\tilde{x}_n - \tilde{x}_{n-1})^2 + (\tilde{y}_n - \tilde{y}_{n-1})^2} \] (11)

The importance of the above part of the controller is coming from the distance \( \lambda \). In other words, we can neglect this part if the distance \( \lambda \) is less than \( \gamma \). Also, we can divide the yaw controller into two sub yaw controllers: one of them represents the controller for the pair at the x-axis and the other is for the pair at the y-axis.

The pair on the x-axis will orient dependently on the commands coming from the yaw controller, which is the Proportional-Differential (PD)
controller output, and they orient in the opposite direction as shown in Fig. 6. In other words, when the distance \(d\) is larger than the specific value \(γ\), the yaw angle will be updated by a new angle \(ψ_d\) which is equal to the angle of the trajectory; otherwise the yaw angle does not be modified. After that, a PD controller is proposed before sending the command to the actuators.

The procedure to update yaw angle is shown as:

\[
\text{If } d > λ \\
\{ \begin{align*}
ψ &= ψ_d \\
\text{Otherwise} \quad ψ &= ψ
\end{align*} \}
\]

where \(ψ_d = \tan^{-1}\left(\frac{Δy}{Δx}\right)\), \(Δy = \bar{y} - y, Δx = \bar{x} - x\)

The pair on the y-axis will be adjusted by a stepper motor that takes command from yaw controller as a constant value \(υ\) which is the step of the stepper motor, otherwise the stepper motor is still with zero rotation. The direction of each stepper motor in this pair is rotating in the same direction of each other to allow the quad-rotor to be capable of rotating.

The equations of the orientation controller are:

\[
R_1 = R_3 = ±(k_{pψ}(ψ_d - ψ) + k_{dψ}(ψ)) \tag{12}
\]

\[
R_2 = R_4 = υ \tag{13}
\]

where \(k_{pψ}\) and \(k_{dψ}\) are the parameters of the PD controller and \(υ\) is the step of the stepper motor.

Now, we can get the forces and moments components as described in the following equations:

\[
F_{Tx} = F_2 \sin R_2 + F_4 \sin R_4 \tag{14}
\]

\[
F_{Ty} = F_1 \sin R_1 - F_3 \sin R_3 \tag{15}
\]

\[
F_{Tz} = \sum_{i=1}^{4} F_i \cos R_i \tag{16}
\]

\[
M_{Tφ} = l_2 F_2 \cos R_2 - l_4 F_4 \cos R_4 \tag{17}
\]

\[
M_{Th} = l_1 F_1 \cos R_1 - l_3 F_3 \cos R_3 \tag{18}
\]

\[
M_{Tψ} = \sum_{i=1}^{4} l_i F_i \sin R_i \tag{19}
\]

\(l_i\) is the length between the propeller and the Center Of Gravity (COG) and \(i=1, 2, 3,\) and \(4\) for each propeller. Observe that the forces and moments are taking effect of the orientation of each propeller by multiplying the forces by \(\sin\) or \(\cos\) of the orientation angle.

b) **Control the speed of each motor**

We control the speed of motor mainly to reach the desired altitude and to get the right direction. This part of controller has three sub-controllers as shown below:

i. **Z-Controller.**

ii. **Roll Controller.**

iii. **Pitch Controller.**

In the following, we will propose a PID controller for each sub-controllers16.

i. **Z-controller.**
The purpose of this controller is to achieve the desired altitude by applying a PID controller on the altitude. The output of this controller should be divided by four to generate the equivalent force on each propeller, hence stabilize the quad-rotor.

The equation of this controller is shown as:

\[ C_i = k_{pz} (z_d - z) + k_{dz} (-\dot{z}) + k_{iz} \int (z_d - z) \]  

(20)

where \( z_d \) is the desired value, and \( k_{pz}, k_{dz}, \text{ and } k_{iz} \) are the parameters of PID controller.

**ii. Roll Controller:**

To move into y-direction, it requires the quad-rotor to roll around the x-axis as shown in Fig. 7. To obtain the roll rotation, we have to generate imbalance forces in the pair on the y-axis. This imbalance force has to rise above the inertial forces opposing the rotation.

The roll angle (\( \phi \)) is approximated as the first-order system of \((y_d - y)\). In other words the roll angle is output of PD controller of \( y \) as following relation:

\[ \phi_d = k_{py} (y_d - y) + k_{dy} (-\dot{y}) \]  

(21)

where \( y_d \) and \( \phi_d \) are the desired values, \( k_{py} \) and \( k_{dy} \) are constant values.

The roll controller will stabilize the roll angle to create the y-motion at its desired value if the gains are well-chosen \(^1\).

**iii. Pitch Controller:**

The pitch controller is the same as the roll controller, except that the controller will be applied on the pitch angle (\( \theta \)) as shown in Fig. 8 which means the rotation about y-axis, to create the motion along the x-axis. This relation is:

\[ \theta_d = k_{pz} (x_d - x) + k_{dx} (-\dot{x}) \]  

(22)

where \( x_d \) and \( \theta_d \) are the desired values, \( k_{pz} \) and \( k_{dx} \) are constant values.

The rotation of the body yaw axis is applied to obtain the components of roll and pitch by using this transformation matrix:

\[ R_{\psi} = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} \]

As mentioned before, if the criterion of using orientation is satisfied, the yawing controller is proposed to get the hovering motion of the quad-rotor. If the criterion is not satisfied, roll and pitch controllers are proposed to keep the hovering motion of the quad-rotor.

Two PD controllers will be applied on roll and pitch angles according to the following relations:
\[ C_2 = k_{p\phi} (\phi_d - \phi) + k_{d\phi} (-\dot{\phi}) \]  
(23)

\[ C_3 = k_{p\theta} (\theta_d - \theta) + k_{d\theta} (-\dot{\theta}) \]  
(24)

where \( C_2 \) and \( C_3 \) are the outputs of the controllers, and \( k_{p\phi}, k_{d\phi}, k_{p\theta}, \) and \( k_{d\theta} \) are the parameters of the controllers.

The output of each controller should be divided by two, with opposite signs to distribute the input controller on each propeller in each pair, to guarantee the balanced forces and these opposite signs mean that each propeller in the pair rotates in the opposite direction of the other.

The equations of the distribution to generate the forces are:

\[ F_1 = \frac{1}{4} C_1 - \frac{1}{2} C_3 \]  
(25)

\[ F_2 = \frac{1}{4} C_1 - \frac{1}{2} C_2 \]  
(26)

\[ F_3 = \frac{1}{4} C_1 + \frac{1}{2} C_3 \]  
(27)

\[ F_4 = \frac{1}{4} C_1 + \frac{1}{2} C_2 \]  
(28)

These forces are sent to the propeller which represents the speed of the propeller that is exactly the speed of the DC motor. In this paper, we focus on the controller to produce the needed forces. Interested readers can refer to Ref. 3, 11, 13 for the information about the DC motor configurations.

The equations of the controller are mainly presented through the Eqs. 12, 13, 20, 23, and 24.

IV. Simulation Results and Analysis

We have applied several trajectories for the vehicle to follow. We can see that the result in Fig. 9 demonstrates the effectiveness of the designed controllers for the motion in the x-y plane. As we can see, the vehicle starts from position (-1.5, 0) and end at the same point (-1.5, 0) through following rectangular path with the constant altitude. In this simulation test, the quad-rotor is commanded to fly from (-1.5,0,1), (-1.5,-0.5,1), (1.5,-0.5,1), (1.5,5.5,1), and (-1.5,5.5,1) and then back to (-1.5,0,1). These results show that the vehicle can track the commanded path and it does not drift away from the path more than 0.4m.

We have approved other paths as shown in Fig. 10 which

\[ \begin{align*}
F_1 &= \frac{1}{4} C_1 - \frac{1}{2} C_3 \\
F_2 &= \frac{1}{4} C_1 - \frac{1}{2} C_2 \\
F_3 &= \frac{1}{4} C_1 + \frac{1}{2} C_3 \\
F_4 &= \frac{1}{4} C_1 + \frac{1}{2} C_2
\end{align*} \]

Figure 9. Trajectory tracking for the vehicle in x-y plane.

Figure 10. Trajectory of the vehicle. The projection of the trajectory in x-y plane is shown in (a). The projection in x-z plane is exposed in (b). The vehicle is changing the altitude from 0 to 6m.
shows the x-y projection of the path in part (a), part (b) shows the x-z projection of the path and the three dimension view is exposed in Fig.11. These figures show the tracking of the vehicle to the desired path in three dimensions. The vehicle follows the path and passes through the commanded points. The use of the orientation controller depends on the distance between the commanded points. In this test, the vehicle is commanded to follow the desired path in the hexagonal form. The vehicle is commanded to fly from (0,-6,2) and then back to (0,-6,2) through the desired path. These results show that the vehicle does not drift away from the path more than 0.5\( m \) at any point during the flight. The trajectory of the vehicle is consistent with the desired path most of the time as shown in Fig. 9. These results also show the effectiveness of the designed controller.

V. Conclusions

We have derived the equations of motion for quad-rotor UAV starting from basic Newton’s second law and proposed a method to control the quad-rotor UAV based on a nonlinear PID controller. This work focused mainly on the control of the vehicle from the side of the orientation of the propeller. We have designed the controller as two parts: the first part controls the orientation of the propeller based on the yaw angle and the second part controls the speed of each propeller. This type of designed controller increases the performance of the quad-rotor in tracking the desired trajectory. As shown in the simulation results, the vehicle was skilled to follow the desired path while the operating points were something like far from each other. This way increases the control actuators which support the reliability of the system. The difficulties of this system are coming from the implementation of the stepper motors with DC motors together from the hardware side and the big size of the code from the software side. The criterion of using the first part depends mainly on the specific value (\( \lambda \)) which gives the indication about the variation between the commanded points. The proposed controller has been implemented in a nonlinear six-degree of freedom simulation model of a quad-rotor UAV and good tracking performances have been obtained through different shapes of commanded trajectories.

References


