ENGR 213: Applied Ordinary Differential Equations

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Instructor and TAs

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- **TAs:**
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  - **Assignments**: Mr. Hesham Kamel; Email: heltaher@gmail.com; EV 13.205; ext 7095

Textbook and Reference

- **Textbook:**
  - ❑ D. G. Zill and M. R. Cullen, *Advanced Engineering Mathematics*, 3rd Edition, Jones & Bartlett Pub, 2006 (ISBN-10: 076374591X; ISBN-13: 978-0763745912)
- **Reference:**
  - ❑ E. Kreyszig, *Advanced Engineering Mathematics*, 9th Edition, Wiley, 2006 (ISBN-10: 0471488852; ISBN-13: 978-0471488859)

Course Outline

- Definitions and Terminology & Initial-Value Problem (Sections 1.1-1.2)
- Differential Equations as Mathematical Models & Solution Curves without a Solution (Sections 1.3 - Section 2.1)
- Separable Variables, Linear Equations & Exact Equations (Sections 2.2-2.4)
- Solutions by Substitutions & a Numerical Solution (Sections 2.5-2.6)
- Linear Models, Nonlinear Models & Systems: Linear and Nonlinear Models (Sections 2.7-2.8)
- Preliminary Theory: Linear Equations (Sections 3.1)
- Reduction of Order & Homogeneous Linear Equations with Constant Coefficients (Sections 3.2-3.3)
- Undetermined Coefficients & Variation of Parameters & Cauchy-Euler Equations & Nonlinear Equations (Sections 3.4-3.7)
- Linear Models & Solving Systems of Linear Equations (Sections 3.8 & 3.11)
- Power Series Solutions of Linear Differential Equations (Section 5.1)
- Numerical Solutions using Euler Methods (Section 6.1)
- Homogeneous Linear Systems & Solution by Diagonalization (Sections 10.1-10.3)
- Nonhomogeneous Linear Systems (Sections 10.4-10.5)

Grading Scheme

• Midterm exams	10% each (during the tutorial)
• Assignments	10%
• Final exam	70%

- If the grade of the final exam is better than the combined mark of the two mid-term examinations then it will carry 90% of the final grade. If the student misses a mid-term test for any reason, including illness, then the final examination will count for 90% of the final grade.
- Any comment or suggestion from you?

Schedule

Lectures:

- Tuesdays and Thursdays, 11:00-13:30, SGW H-620

Tutorials:

- Tuesdays and Thursdays, 08:45-10:25, SGW H-620

Office Hours:

- Tuesdays and Thursdays, 14:30 - 15.30pm, EV 4-109

Course Webpage:

- http://users.encs.concordia.ca/~ymzhang/courses/ENGR213.html
- See also from Moodle later

Exam Schedule:

- Midterm I - suggested: Tue., 5/22/07 during the tutorial
- Midterm II - suggested: Tue., 6/05/07 during the tutorial

ENGR213: Applied Ordinary Differential Equations

Chapter 1

Introduction

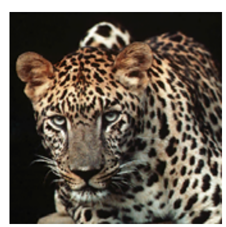
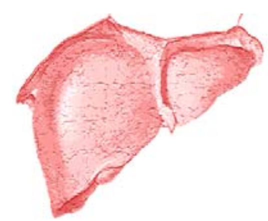
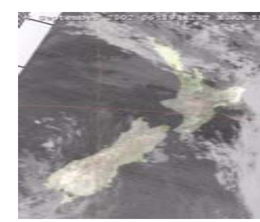
- Where Do Ordinary Differential Equations (ODEs) Arise?
- Definitions and Terminology (Section 1.1)
- Initial-Value Problems (Section 1.2)
- DEs as Mathematical Models (Section 1.3)

Where Do ODEs Arise?

- All branches of Engineering
- Economics
- Biology and Medicine
- Chemistry, Physics etc

Anytime you wish to find out how something changes with time (and sometimes space)

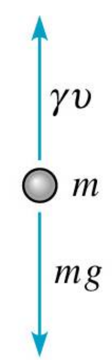



Example 1: Free Fall

- Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.
- Variables: time t, velocity v
- Newton's 2nd Law: $F = ma = m(\mathrm{d}v/\mathrm{d}t)$ ←net force
- Force of gravity: $F = mg$ ←downward force
- Force of air resistance: $F = \gamma v$ ←upward force
- Then

$$m\frac{dv}{dt} = mg - \gamma v$$

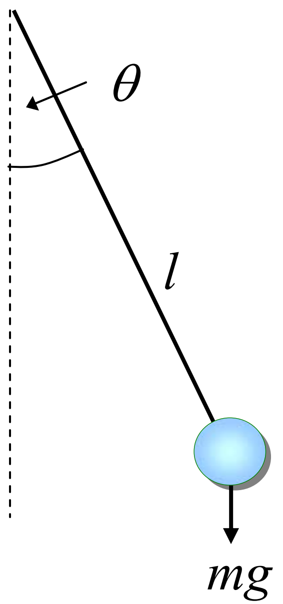


First-order derivative

Example 2: Swinging of a Pendulum

Newton's 2nd law for a rotating object:

- Moment of inertia · Angular acceleration = Net external torque



$$ml^2 \cdot \frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

rearrange and divide through by ml^2

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$

where $\omega^2 = \frac{g}{l}$

2nd-order derivative

Questions:

1) How to find these equations?

2) How to solve these equations?

Objectives of this course!

Objectives

- The main purpose of this course is to discuss properties of solutions of differential equations, and to present methods of finding solutions of these differential equations.
- To provide a framework for this discussion, in the first part of lecture we will give definitions of differential equations and provide several ways of classifying differential equations.

ENGR213: Applied Ordinary Differential Equations

Chapter 1

Introduction

- Where Do Ordinary Differential Equations (ODEs) Arise?
- *Definitions and Terminology (Section 1.1)*
- Initial-Value Problems (Section 1.2)
- DEs as Mathematical Models (Section 1.3)

Definition and Classification

❑ Definition 1.1: Differential Equation

An equation containing the *derivatives* of one or more *dependent variables*, with respect to one or more *independent variables*, is said to be a **differential equation** (**DE**).

❑ Classification by Type

1) Ordinary Differential Equation (ODE): If an equation contains only *ordinary* derivatives of one or more dependent variables w.r.t. a single independent variable, it is said to be an ODE.

   Example: $\frac{dy}{dx} + 5y = e^x$

2) Partial Differential Equation (PDE): An equation involving the *partial* derivatives of one or more dependent variables w.r.t. two or more independent variable is called a PDE.

   Example: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$

Definition and Classification

Classification by Order

The order of a differential equation is just the order of the highest derivative in the equation.

1) First-order ODE
2) Second-order ODE
3) Higher-order ODE

Example:

$$\frac{dy}{dx}+5y=e^{x}$$ → 1st order

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$$ → 2nd order

$$\frac{dx}{dt}=x\frac{d^3x}{dt^3}$$ → 3rd order

Definition and Classification

- **Classification by Linearity**
  - The important issue is how the unknown y appears in the equation. A linear equation involves the dependent variable (y) and its derivatives by themselves. There must be no "unusual" nonlinear functions of y or its derivatives.
  - A linear equation must have constant coefficients, or coefficients which depend on the independent variable (x, or t). If y or its derivatives appear in the coefficient the equation is non-linear.

Definition and Classification

❑ Classification by Linearity

- *Example*:

$$\frac{dy}{dx} + y = 0$$ is linear

$$\frac{dy}{dx} + y^2 = 0$$ is non-linear

$$\frac{dy}{dx} + x^2 = 0$$ is linear

$$y\frac{dy}{dx} + x^2 = 0$$ is non-linear

Leibniz's notation

Prime notation

How about following equations in terms of order and linearity?

$$(y - x)dx + 4xdy = 0, \; y'' - 2y' + y = 0, \; \frac{d^3y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$$

Definition and Classification

❑ Classification by Linearity

- *Summary*:

Linear	Non-linear
$2y$	y^2 or $\sin(y)$
$\frac{dy}{dt}$	$y\frac{dy}{dt}$
$(2+3\sin t)y$	$(2-3y^2)y$
$t\frac{dy}{dt}$	$\left(\frac{dy}{dt}\right)^2$

Definition and Classification

- **Classification by Linearity**
  - *Special Property*:

If a linear homogeneous ODE has solutions:

$$y = f(t) \quad \text{and} \quad y = g(t)$$

then:

$$y = a \times f(t) + b \times g(t)$$

where a and b are constants

is also a solution.

Definition and Classification

- **Classification by Homogeniety**
  - Put all the terms of the equation which involve the dependent variable on the LHS (left-hand-side).
  - **Homogeneous:** If there is nothing left on the RHS (right-hand-side), the equation is homogeneous (unforced or free)
  - **Nonhomogeneous:** If there are terms involving x (or constants) - but not y - left on the RHS the equation is nonhomogeneous (forced)

Definition and Classification

- **Classification by Initial Value/Boundary Value Problems**
  - **Initial Value Problem:** Problems that involve independent variable x (or time t) are represented by an ODE together with *initial values*.
  - **Boundary Value Problem:** Problems that involve space (just one dimension) are also governed by an ODE but what is happening at the *ends of the region* of interest has to be specified as well by boundary conditions.

Definition and Classification

❑ Examples

$$\frac{dv}{dt} = g$$

$$v(0) = v_0$$

- 1st order
- Linear
- Nonhomogeneous
- Initial value problem

$$\frac{d^2 M}{dx^2} = w$$

$$M(0) = 0$$

and

$$M(l) = 0$$

- 2nd order
- Linear
- Nonhomogeneous
- Boundary value problem

Definition and Classification

- **Examples**

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$

$$\theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0$$

- 2nd order
- Nonlinear
- Homogeneous
- Initial value problem

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0$$

$$\theta(0) = \theta_0, \quad \frac{d\theta}{dt}(0) = 0$$

- 2nd order
- Linear
- Homogeneous
- Initial value problem

Solution of an ODE

- **Definition 1.2: Solution of an ODE**

  Any function φ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when submitted into an nth-order ODE reduces the equation to an *identity*, is said to be a **solution** of the equation on the interval.

- **Interval of Definition**

  The interval I in Definition 1.2 is variously called the *interval of definition*, the *interval of existence* etc and can be an open interval (a,b), a closed interval $[a,b]$, an infinite interval (a, ∞), and so on.

Solution of an ODE

❑ **Example 1** Verification of a Solution

Verify that the indicated function is a solution of the given DE on the interval $(-\infty, \infty)$

(a) $dy/dx = xy^{1/2}$; $y = x^4/16$ Leibniz's notation

(b) $y'' - 2y' + y = 0$; $y = xe^x$ Prime notation

Solution: Verifying whether each side of the equation is the same for every real number x in the interval

(a) From LHS: $$\frac{dy}{dx} = 4 \cdot \frac{x^3}{16} = \frac{x^3}{4}$$

RHS: $$xy^{1/2} = x \cdot \left(\frac{x^4}{16}\right)^{1/2} = x \cdot \frac{x^2}{4} = \frac{x^3}{4}$$

Note: each side of the equation is the same for every x.

(b) Solve by yourself as exercise

Solution of an ODE

- **Example 1** Verification of a Solution

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$

for every real number x

LHS: $y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$

RHS: 0 ❑

- **Trivial solution**: A solution of a differential equation that is identically zero on an interval I is said to be a *trivial solution*.

Solution of an ODE

- **Solution Curve**

  The graph of a solution φ of an ODE is called a **solution curve**. Since φ is a differential function, it is continuous on its interval *I*.

- **Example 2: Function vs. Solution**

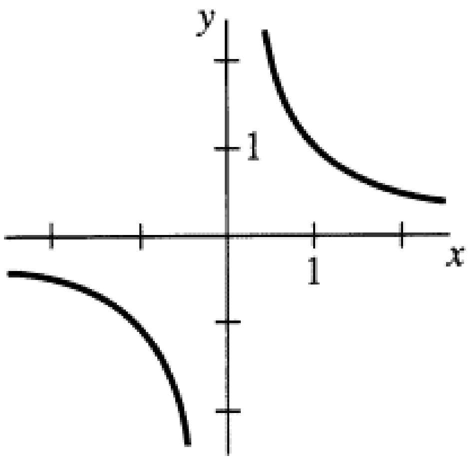


Fig 1.1 (a) Function $y = 1/x$, $x \neq 0$

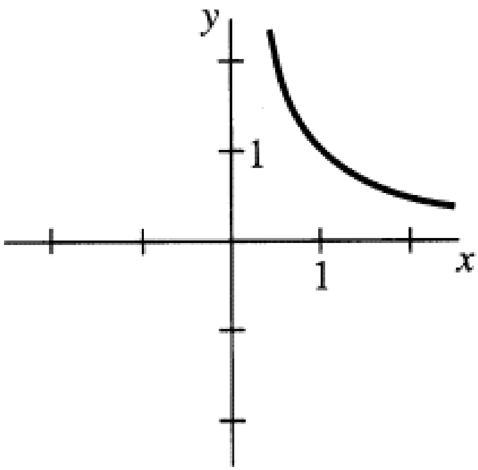


Fig 1.1 (b) Solution $y = 1/x$, $(0, \infty)$

Solution of an ODE

❑ Further example

Solution Curves

The ODE $y' = dy/dx = \cos x$ can be solved directly by integration on both sides. Indeed, using calculus, we obtain $y = \int \cos x \, dx = \sin x + c$, where c is an arbitrary constant. This is a ***family of solutions.*** Each value of c, for instance, 2.75 or 0 or -8, gives one of these curves. Figure 2 shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.

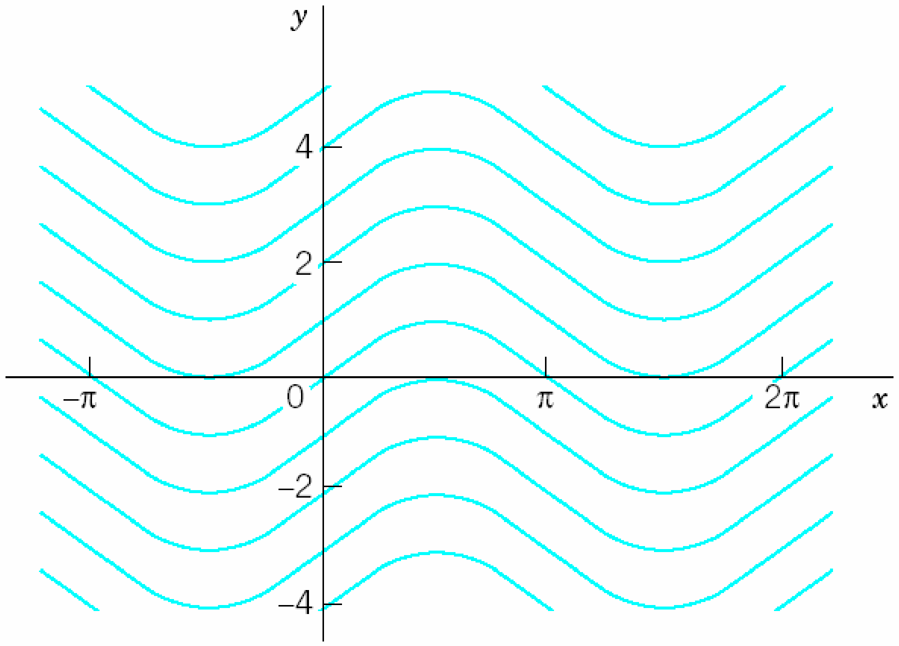


Fig. 2. Solutions $y = \sin x + c$ of the ODE $y' = \cos x$

Solution of an ODE

- **Explicit and Implicit Solutions**

  1) *Explicit solution*: A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an *explicit solution.*

  2) *Implicit solution*: see definition below

- **Definition 1.3: Implicit Solution of an ODE**

  A relation $G(x, y) = 0$ is said to be an *implicit solution* of an ODE on an interval I provided there exist at least one function φ that satisfies the relation as well as the differential equation on I.

Solution of an ODE

- **Example 3** Verification of an Implicit Solution

The relation $x^2 + y^2 = 25$ is an implicit solution of the following DE on the interval $-5 < x < 5$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad (8)$$

Solution: By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \quad \text{or} \quad 2x + 2y\frac{dy}{dx} = 0$$

Solving above equation gives (8); Solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$, or

$$y = \phi_1(x) = \sqrt{25 - x^2} \qquad y = \phi_2(x) = -\sqrt{25 - x^2}$$

satisfy the relation $x^2 + \phi_1{}^2 = 25$ and $x^2 + \phi_2{}^2 = 25$ and are explicit solution defined on the interval.

Solution of an ODE

The <u>solution curve</u> given in Figs. 1.2(b) and 1.2(c) are segments of the graph of the implicit solution in Fig. 1.2(a).

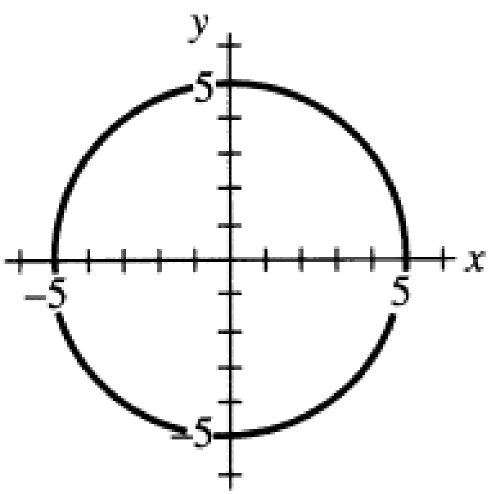


Fig. 1.2(a) IS

$x^2 + y^2 = 25$

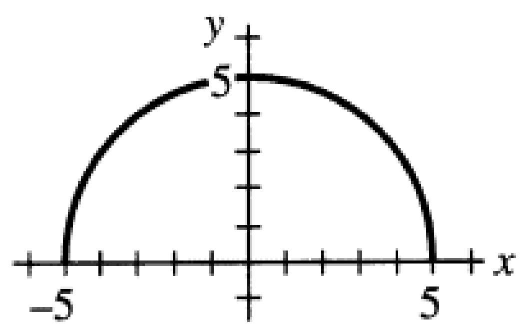


Fig. 1.2(b) ES

$y_1 = \sqrt{25 - x^2}, -5 < x < 5$

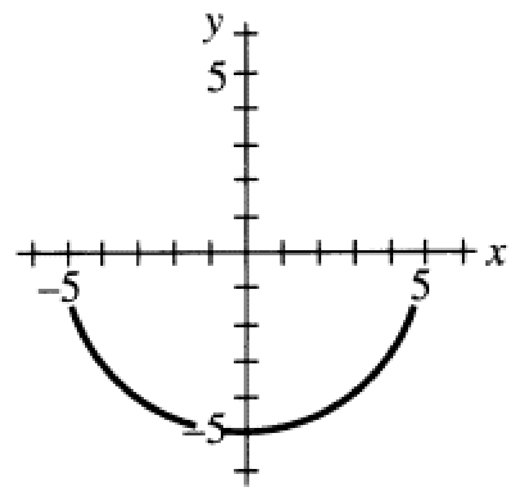


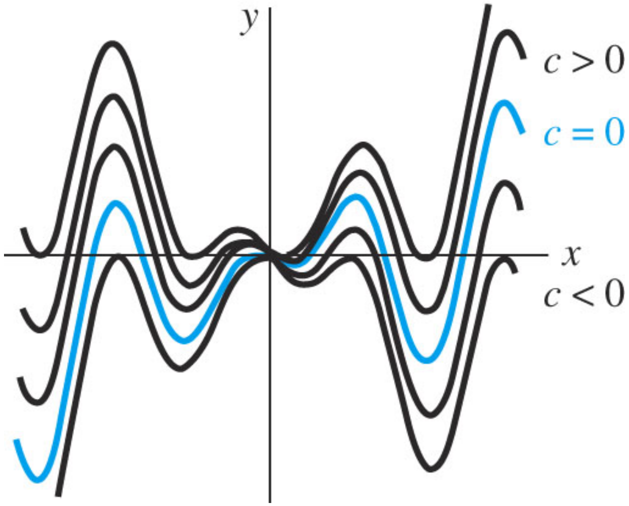
Fig. 1.2(c) ES

$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

Solution of an ODE

❑ Families of Solutions

1) A single DE can possess an infinite number of solutions to the unlimited number of choices for the parameter(s) – a family of solutions.



❑ Particular Solution

A solution of a DE that is free of arbitrary parameters is called a **particular solution**.

Ex: the one-parameter family $y = cx - x\cos(x)$ is an explicit solution of the linear 1st-order equation $xy' - y = x^2\sin(x)$ on the interval $(-\infty, \infty)$. $y = -x\cos(x)$ is a particular solution when $c=0$.

❑ Singular Solution

An extra solution that cannot be obtained by specializing any of the parameters in the family of solutions.

Solution of an ODE

Example:

$y = x^4/16$ and $y = 0$ are solutions of $dy/dx = xy^{1/2}$ on $(-\infty, \infty)$. The DE possesses the one-parameter family of solutions $y = (x^2/4 + c)^2$. When $c = 0$, the resulting *particular solution* is $y = x^4/16$. But the *trivial solution* $y = 0$ is a *singular solution* since it is not a member of the solution family; there is no way of assigning a value to the constant c to obtain $y = 0$.

- **General Solution**

If *every* solution of an nth-order ODE $F(x, y, y', \cdots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i=1,2,\ldots,n$, we say that the family is the general solution of the DE.

Ex: 1.1-4: Using Differnt Symbols

The function $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0$$

For $x=c_1\cos 4t$, the 1st,2nd derivatives w.r.t. t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting x'' and x then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0$$

Similarly, for $x=c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_1 \sin 4t + 16(c_1 \sin 4t) = 0$$

Finally, it is straightforward to verify that the linear combination of solutions for the two-parameter family $x=c_1\cos 4t + c_2\sin 4t$ is also a solution of the DE. ❑

Solution of an ODE

❑ Systems of Differential Equations

A system of ODEs is *two or more equations* involving the derivatives of *two or more unknown functions* of a *single independent* variables.

For example, if x and y denote dependent variables and t the independent variable, then a system of two first-order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y) \qquad (9)$$

A **solution** of a system such as (9) is a pair of differentiable functions $x = \phi_1(t),\ y = \phi_2(t)$ defined on a common interval I that satisfy each equation of the system on this interval.

ENGR213: Applied Ordinary Differential Equations

Chapter 1

Introduction

- Where Do Ordinary Differential Equations (ODEs) Arise?
- Definitions and Terminology (Section 1.1)
- *Initial-Value Problems (Section 1.2)*
- DEs as Mathematical Models (Section 1.3)

1.2 Initial-Value Problems (IVPs)

On some interval I containing x_0, the problem

Solve: $$\frac{d^n y}{dx^n} = f(x, y, y', \cdots, y^{(n-1)})$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1} \qquad (1)$$

where $y_0,\ y_1,\ \cdots,\ y_{n-1}$ are arbitrarily specified real constants, is called an **initial-value problem (IVP)**.

The value of $y(x)$ and its first n-1 derivatives at a single point x_0:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1}$$

are called **initial conditions**.

First-Order IVPs

Solve: $\dfrac{dy}{dx} = f(x,y)$

Subject to: $y(x_0) = y_0$ (2)

For (2), we are seeking a solution of the differential equation on an interval *I* containing x_0 so that a solution curve pass through the prescribed point (x_0, y_0). See Fig. 1.7.

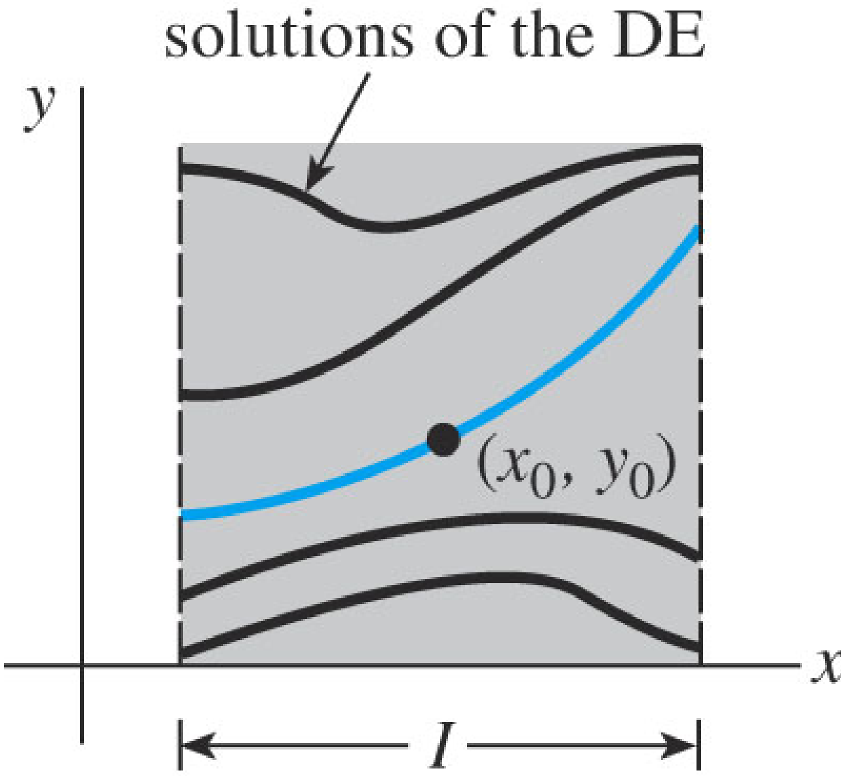


Fig. 1.7

Second-Order IVPs

Solve: $$\frac{d^2y}{dx^2} = f(x, y, y')$$ (3)

Subject to: $y(x_0) = y_0, y'(x_0) = y_1$

For (3), we want to find a solution of the differential equation whose graph not only passes through (x_0, y_0) but passes through so that the *slope* of the curve at this point is y_1. See Fig. 1.8.

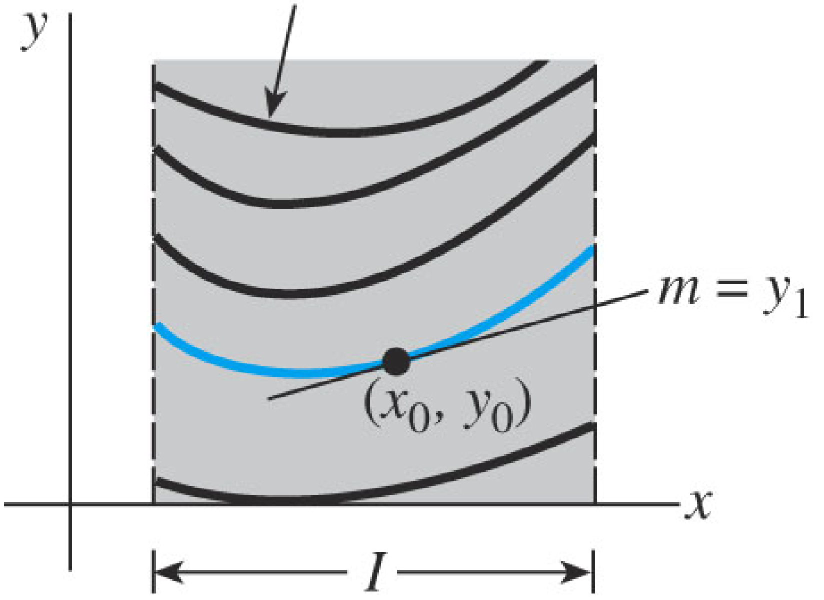


Fig. 1.8

Example 1.2-3: 2nd-Order IVP

In Ex. 4 of Sect. 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$

is a two-parameter family of solution of $x'' + 16x = 0$

Find a solution of the initial-value problem.

$$x'' + 16x = 0, \qquad x\left(\frac{\pi}{2}\right) = -2, \qquad x'\left(\frac{\pi}{2}\right) = 1 \qquad (4)$$

Solution:

Apply $x(\pi/2) = -2$ to $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$

Since $\cos 2\pi = 1$, $\sin 2\pi = 0$, we find that $c_1 = -2$

Next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2\cos 4t + c_2 \sin 4t$. Differentiating & setting $t = \pi/2, x' = 1$ gives $8\sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = 1/4$. Hence

$$x = -2\cos 4t + \frac{1}{4}\sin 4t$$

Existence of a Unique Solution

Two fundamental questions:

- *Does a solution of the problem exist?*
- *If a solution exists, is it unique?*

For an IVP such as (2), we think about:

Existence
- *Does the DE $dy/dx = f(x, y)$ possess solutions?*
- *Do any of solution curves pass through the point* (x_0, y_0) ?

Uniqueness
- *When can we be certain that there is precisely one solution curve passing through the point* (x_0, y_0)?

Ex. 4: Several Solutions in an IVP

Functions $y = 0$ and $y = x^4/16$ both satisfy DE $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, and so the initial-value problem has at least two solutions, as shown in Fig. 1.11, the graphs of both functions pass through the same point (0, 0).

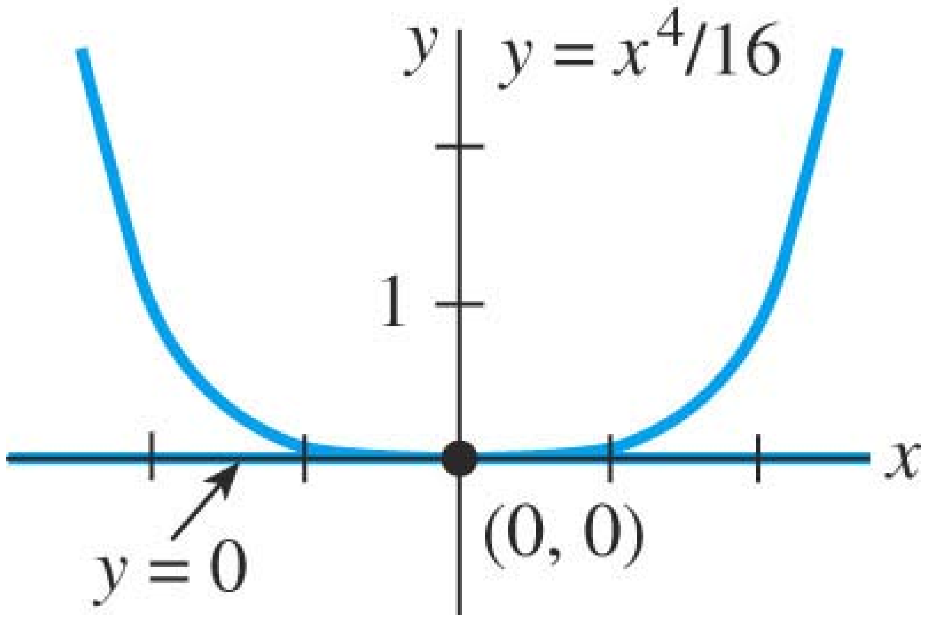


Fig.1.11 Two solutions of the same IVP

Theorem 1.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b, c \le y \le d$ It contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $x_0 - h < x < x_0 + h, h > 0$ contained in $a \le x \le b$: and a unique function $y(x)$ defined on I_0 that is a solution of the initial-value problem (2) (see Fig. 1.12).

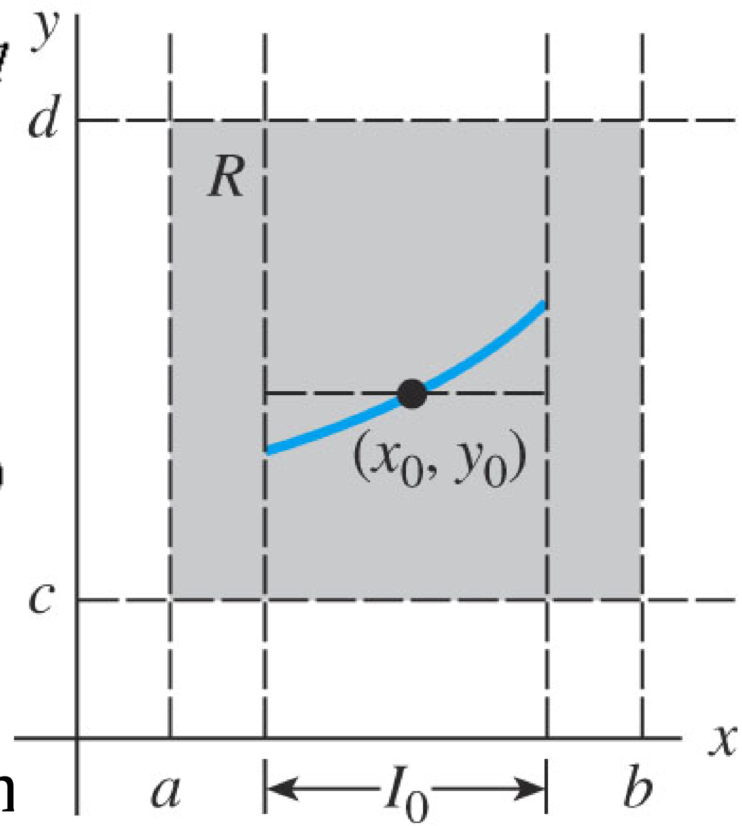


Fig.1.12 Rectangular region R

Example 1.2-3: Revisited

From Ex. 1.2-3, we saw that the DE $dy/dx = xy^{1/2}$ possesses at least two solutions whose graphs pass through (0, 0). Inspection of the functions

$$f(x, y) = xy^{1/2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane by $y > 0$. Hence Theorem 1.1 enables us to conclude that through any point (x_0, y_0), $y_0 > 0$, in the upper half-plane there is some interval centered at x_0 on which the given DE has a unique solution.

For example, even without solving it we know that there exists some interval centered at 2 on which the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$, has a unique solution. ❑

ENGR213: Applied Ordinary Differential Equations

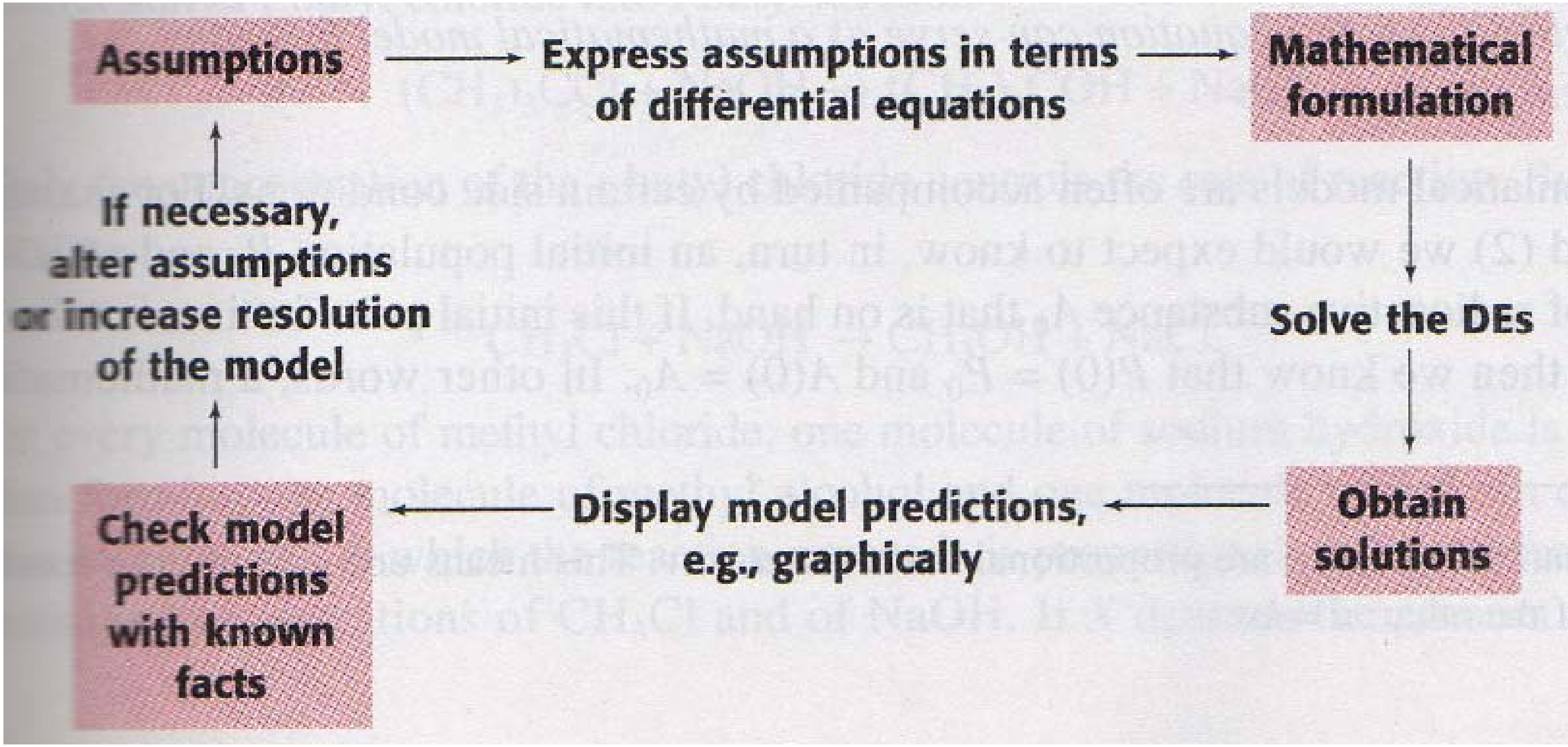
Chapter 1

Introduction

- Where Do Ordinary Differential Equations (ODEs) Arise?
- Definitions and Terminology (Section 1.1)
- Initial-Value Problems (Section 1.2)
- *DEs as Mathematical Models (Section 1.3)*

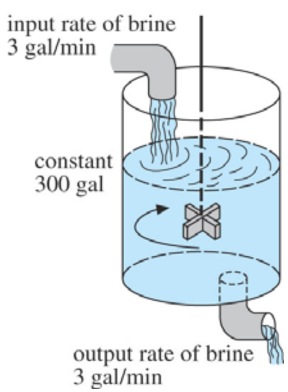
1.3 DEs as Mathematical Models

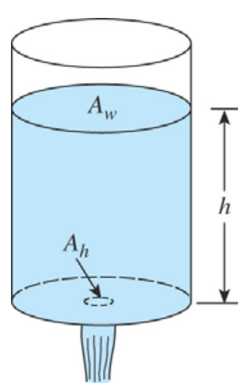
A differential equation that describes a physical process is often called a **mathematical model**.

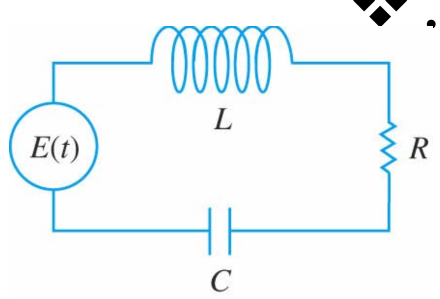


1.3 DEs for Different Applications

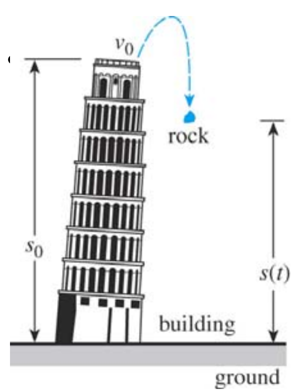
- Population Dynamics
- Radioactive Decay
- Newton's Law of Cooling/Warming
- Spread of Disease
- Chemical Reactions
- Mixtures
- Draining a Tank
- Series Circuits
- Falling Bodies
- Falling Bodies and Air Resistance
- A Slipping Chain
- Suspended Cables
- ……

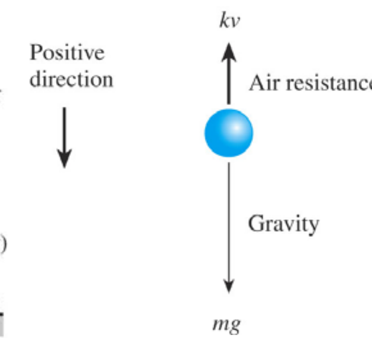






(a) *LRC*-series circuit





Reading and Exercise

- **Reading**
  - ❑ Sections 1.1 & 1.2
- **Assignment** (Due on the following week at Thursday's tutorial)
  - ❑ **Section 1.1**: 1,2,10,3,5,6,8,11,13,14,23,24. (3rd edition)
  - ❑ **Section 1.2**: 7,9,11,12,17,18. (3rd edition)
  - ❑ **Section 1.3**: 10,13. (3rd edition)

Your any questions, suggestions or comments are welcome