FP8-4: Estimation and Sensor Information Fusion

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FP8-4: Estimation and Sensor Information Fusion

Lecture 1

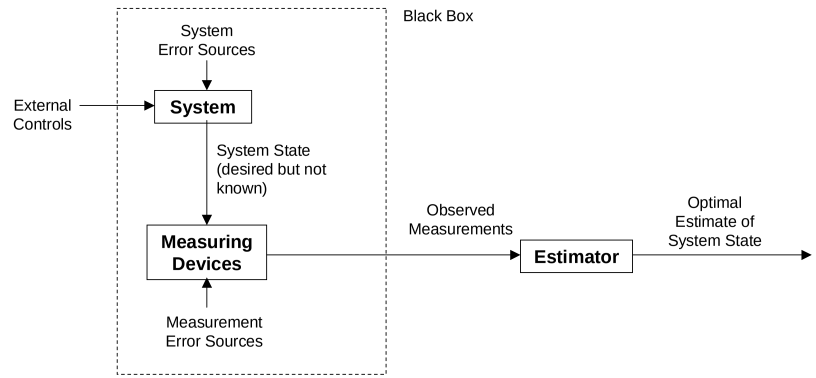
Review on Kalman Filters

- Course outline
- What is Kalman filter (KF) and applications of KF?
- The discrete-time Kalman filter (DKF)
- The continuous-time Kalman filter (CKF)
- The Kalman filters for nonlinear systems

Course Outline

1. Review on Kalman filter and its variants
2. Multiple-model based Kalman filters
3. Simultanuous state and parameter estimation: Two-stage Kalman filter
4. Introduction to sensor information fusion (I)
5. Introduction to sensor information fusion (II) and course summary

The Problem – Why do we need Kalman Filters?



- System state cannot be measured directly
- Need to estimate "optimally" from measurements

All Roads Lead from Gauss 1809

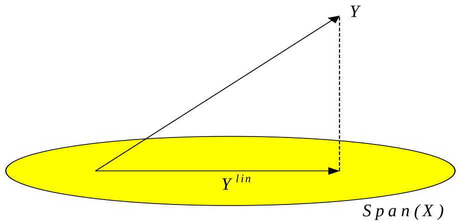
" … since all our measurements and observations are nothing more than approximations to the truth, the same must be true of all calculations resting upon them, and the highest aim of all computations made concerning concrete phenomenon must be to approximate, as nearly as practicable, to the truth. But this can be accomplished in no other way than by suitable combination of more observations than the number absolutely requisite for the determination of the unknown quantities. This problem can only be properly undertaken when an approximate knowledge of the orbit has been already attained, which is afterwards to be corrected so as to satisfy all the observations in the most accurate manner possible."

- From Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections, Gauss, 1809.

Estimation Basics

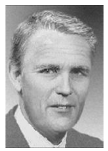
- Problem statement
  - Observation random variable X (Given)
  - Target random variable Y (Unknown)
  - Joint probability density $f(x,y)$ (Given)
- What is the best estimate $y^{opt}=g(x)$ which minimizes the *expected mean square error* between y^{opt} *and* y ?
- Answer: Conditional Mean $g(x) = E(Y|X=x)$
- Estimate $g(x)$ can be potentially nonlinear and unavailable in closed-form.
- When X and Y are jointly Gaussian $g(x)$ is linear.
- What is the best linear estimate $y^{lin}=Wx$ which minimizes the mean square error ?

Wiener Filter 1940



- Wiener-Hopf Solution: $W = R_{YX} (R_{xx})^{-1}$
- Involves matrix inversion
- Applies only to stationary processes
- Not amenable for online recursive implementation.

Kalman Filter 1960



STATE-SPACE MODEL

Process Model : $x_{k+1} = A_k x_k + w_k$

Measurement Model : $y_{k+1} = M_{k+1} x_{k+1} + v_{k+1}$

- The estimate can be obtained recursively.
- Can be applied to non-stationary processes.
- If measurement noise and process noise are white and Gaussian, then the filter is "optimal".
  - Minimum variance unbiased estimate
- In the general case, the Kalman filter is the best linear estimator among all linear estimators.
- It can be extended to nonlinear estimation – extended Kalman filter (EKF)

Why Do We Need KF ?

- A prediction/correction algorithm used for *state estimation* under *stochastic* environment
- KF is used for:
  - *Estimation* of a value where measurements are made in noisy environments (noisy input is converted to less noisy data)
  - *Prediction* of time varying variable based on a linear state model (based on previous measurements)
  - *Data fusion* (used for obtaining an estimate of a value whose measurement is obtained from different sources)
  - …

How Does It Do It ?

- The Kalman filter iterates between two steps
  - Time Update (Predict)
    - Project current (k-1) state and covariance forward to the next time step (k), that is, compute the next a priori estimates.
  - Measurement Update (Correct)
    - Update the a priori quantities using current time step noisy measurements, that is, compute the a posteriori estimates.

$$\hat{x}_k = \hat{x}_k^- + K_k\left(z_k - M_k\hat{x}_k^-\right)$$

- Choose K_k (Kalman gain) to minimize error covariance

Common Applications

- Tracking missiles
  - Tracking moving objects
- GPS
- Computer vision
  - Extracting lip motion from video
- Data fusion/integration
  - Integration of spatio-temporal video segments
- Robotics
  - Robust estimation and sensor data noise reduction
- State and parameter estimation for monitoring, fault diagnosis and control
- Data smoothing and curve fitting
- ……

Linear Plant and Measurement Models

TABLE 4.1 Linear Plant and Measurement Models

Model	Continuous Time	Discrete Time	Equation Number
Plant	$\dot{x}(t) = F(t)x(t) + w(t)$	$x_k = \Phi_{k-1}x_{k-1} + w_{k-1}$	(4.1)
Measurement	$z(t) = H(t)x(t) + v(t)$	$z_k = H_k x_k + v_k$	(4.2)
Plant noise	$E\langle w(t)\rangle = 0$	$E\langle w_k\rangle = 0$	(4.3)
	$E\langle w(t)w^{T}(s)\rangle = \delta(t-s)Q(t)$	$E\langle w_k w_i^{T}\rangle = \Delta(k-i)Q_k$	(4.4)
Observation noise	$E\langle v(t)\rangle = 0$	$E\langle v_k\rangle = 0$	
	$E\langle v(t)v^{T}(s)\rangle = \delta(t-s)R(t)$	$E\langle v_k v_i^{T}\rangle = \Delta(k-i)R_k$	(4.5)

TABLE 4.2 Dimensions of Vectors and Matrices in Linear Model

Symbol	Dimensions	Symbol	Dimensions
x, w	$n \times 1$	Φ, Q	$n \times n$
z, v	$\ell \times 1$	H	$\ell \times n$
R	$\ell \times \ell$	Δ, δ	scalar

Estimation Problem - Objective

Objective: To find an estimation of the n state vector x_k represented by $\hat{x}_k$, a linear function of the measurements $z_i, \ldots, z_k$, that minimizes the weighted mean-squared error

$$E[x_k - \hat{x}_k]^T [x_k - \hat{x}_k]$$

where M is any symmetric nonnegative-definite weighting matrix.

Solution?

The Discrete-Time Kalman Filter

A representation from GA01

TABLE 4.3 Discrete-Time Kalman Filter Equations

System dynamic model:

$$x_k = \Phi_{k-1} x_{k-1} + w_{k-1}$$
$$w_k \sim \mathcal{N}(0, Q_k)$$

Measurement model:

$$z_k = H_k x_k + v_k$$
$$v_k \sim \mathcal{N}(0, R_k)$$

Initial conditions:

$$E\langle x_0 \rangle = \hat{x}_0$$
$$E\langle \tilde{x}_0 \tilde{x}_0^T \rangle = P_0$$

Independence assumption:

$$E\langle w_k v_j^T \rangle = 0 \quad \text{for all } k \text{ and } j$$

State estimate extrapolation (Equation 4.25):

$$\hat{x}_k(-) = \Phi_{k-1} \hat{x}_{k-1}(+)$$

Error covariance extrapolation (Equation 4.26):

$$P_k(-) = \Phi_{k-1} P_{k-1}(+) \Phi_{k-1}^T + Q_{k-1}$$

State estimate observational update (Equation 4.21):

$$\hat{x}_k(+) = \hat{x}_k(-) + \overline{K}_k [z_k - H_k \hat{x}_k(-)]$$

Error covariance update (Equation 4.24):

$$P_k(+) = [I - \overline{K}_k H_k] P_k(-)$$

Kalman gain matrix (Equation 4.19):

$$\overline{K}_k = P_k(-) H_k^T [H_k P_k(-) H_k^T + R_k]^{-1}$$

$$\hat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(+),$$
$$P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T,$$
$$\hat{x}_k(+) = \hat{x}_k(-) + \overline{K}_k(z_k - H_k\hat{x}_k(-)),$$
$$\overline{K}_k = P_k(-)H_k^T(H_kP_k(-)H_k^T + R)^{-1},$$
$$P_k(+) = P_k(-) - \overline{K}_kH_kP_k(-),$$

The Discrete-Time Kalman Filter

A representation from BLK01

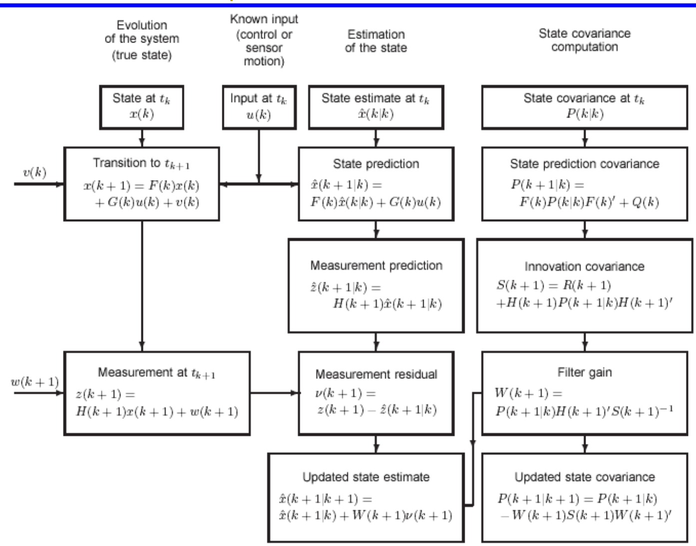


Figure 5.2.4-1: One cycle in the state estimation of a linear system.

The Discrete-Time Kalman Filter

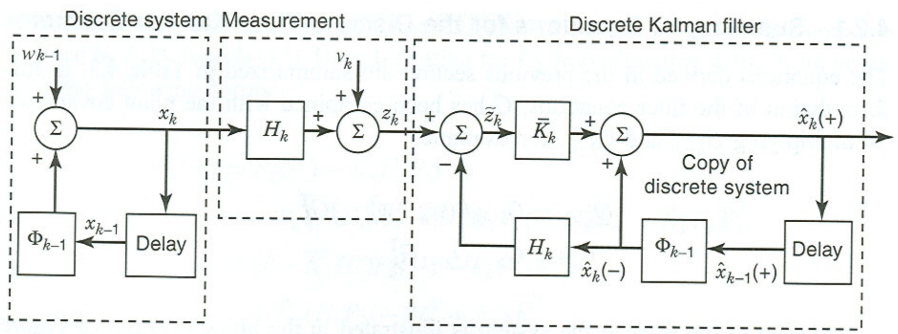


Fig. 4.1 Block diagram of system, measurement model, and discrete-time Kalman filter.

The Kalman Filter Cycle

Time Update ("Predict")

(1) Project the state ahead

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1}$$

(2) Project the error covariance ahead

$$P_k^- = AP_{k-1}A^T + Q$$

Measurement Update ("Correct")

(1) Compute the Kalman gain

$$K_k = P_k^- H^T (HP_k^- H^T + R)^{-1}$$

(2) Update estimate with measurement z_k

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

(3) Update the error covariance

$$P_k = (I - K_k H)P_k^-$$

Initial estimates for $\hat{x}_{k-1}$ and P_{k-1}

Sequence of Operation of DKF

$$\hat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(+),$$
$$P_k(-) = \Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T,$$
$$\hat{x}_k(+) = \hat{x}_k(-) + \overline{K}_k(z_k - H_k\hat{x}_k(-)),$$
$$\overline{K}_k = P_k(-)H_k^T(H_kP_k(-)H_k^T + R)^{-1},$$
$$P_k(+) = P_k(-) - \bar{K}_kH_kP_k(-),$$

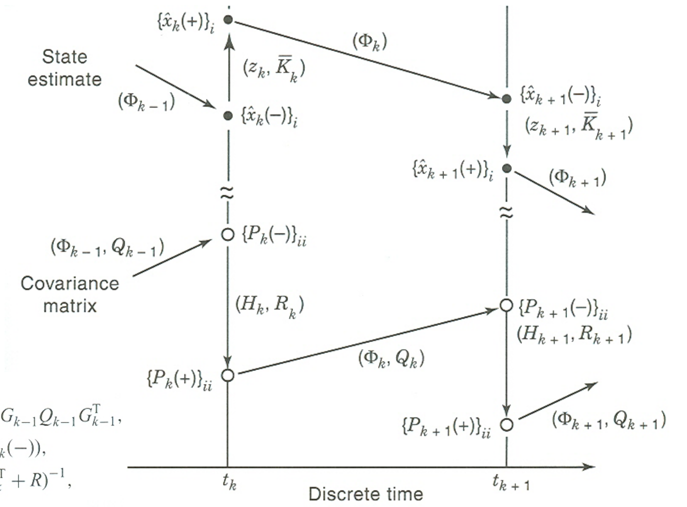
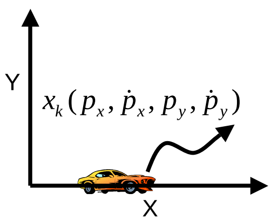


Fig. 4.2 *Representative sequence of values of filter variables in discrete time.*

A Simple Example - Tracking



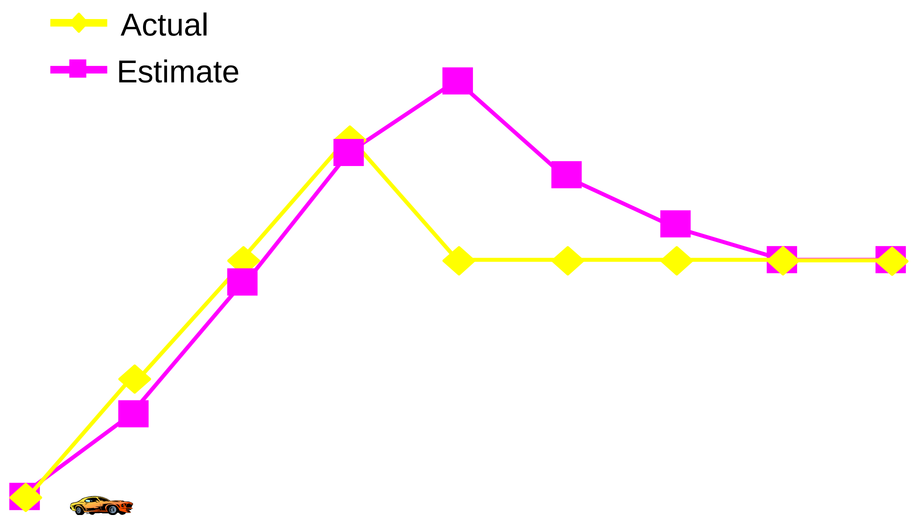
$$x_{k+1} = Fx_k + w_k$$

$$\begin{bmatrix} p_x \\ \dot{p}_x \\ p_y \\ \dot{p}_y \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & \delta t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ \dot{p}_x \\ p_y \\ \dot{p}_y \end{bmatrix}_k + w_k$$

$$z_k = Hx_k + v_k$$

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ \dot{p}_x \\ p_y \\ \dot{p}_y \end{bmatrix}_k + v_k$$

A Simple Example – Estimated vs. Actual



The Discrete-Time Kalman Filter

An Example

EXAMPLE 4.1 Let the system dynamics and observations be given by the following equations:

$$x_k = x_{k-1} + w_{k-1}, \qquad z_k = x_k + v_k,$$
$$E\langle v_k \rangle = Ew_k = 0,$$
$$E\langle v_{k_1} v_{k_2} \rangle = 2\Delta(k_2 - k_1), \quad E\langle w_{k_1} w_{k_2} \rangle = \Delta(k_2 - k_1),$$
$$z_1 = 2, \qquad z_2 = 3,$$
$$E\langle x(0) \rangle = \hat{x}_0 = 1,$$
$$E\langle [x(0) - \hat{x}_0][x(0) - \hat{x}_o]^T \rangle = P_0 = 10.$$

The objective is to find $\hat{x}_3$ and the steady-state covariance matrix P_∞. One can use the equations in Table 4.3 with

$$\Phi = 1 = H, \qquad Q = 1, \qquad R = 2,$$

The Discrete-Time Kalman Filter

An Example (cont) – more examples in overheader and MATLAB

for which

$$\boxed{P_k(-) = P_{k-1}(+) + 1} \; ,$$
$$\overline{K}_k = \frac{P_k(-)}{P_k(-) + 2} = \frac{P_{k-1}(+) + 1}{P_{k-1}(+) + 3},$$
$$P_k(+) = \left[1 - \frac{P_{k-1}(+) + 1}{P_{k-1}(+) + 3}\right](P_{k-1}(+) + 1),$$
$$\boxed{P_k(+) = \frac{2(P_{k-1}(+) + 1)}{P_{k-1}(+) + 3}} \; .$$
$$\boxed{\hat{x}_k(+) = \hat{x}_{k-1}(+) + \overline{K}_k(z_k - \hat{x}_{k-1}(+))} \; .$$

Let

$$P_k(+) = P_{k-1}(+) = P \quad \text{(steady-state covariance)},$$
$$P = \frac{2(P+1)}{P+3},$$
$$P^2 + P - 2 = 0,$$
$$P = 1, \quad \text{positive-definite solution.}$$

For $k = 1$

$$\hat{x}_1(+) = \hat{x}_0 + \frac{P_0 + 1}{P_0 + 3}(2 - \hat{x}_0) = 1 + \frac{11}{13}(2 - 1) = \frac{24}{13}$$

Following is a table for the various values of the Kalman filter:

k	$P_k(-)$	$P_k(+)$	$\overline{K}_k$	$\hat{x}_k(+)$
1	11	$\frac{22}{13}$	$\frac{11}{13}$	$\frac{24}{13}$
2	$\frac{35}{13}$	$\frac{70}{61}$	$\frac{35}{61}$	$\frac{153}{61}$

The Discrete-Time Kalman Filter

4.14.2 Important Equations to Remember

Kalman Filter. The *discrete-time model* for a linear stochastic system has the form

$$x_k = \Phi_{k-1} x_{k-1} + G_{k-1} w_{k-1},$$
$$z_k = H_k x_k + v_k,$$

where the zero-mean uncorrelated Gaussian random processes $\{w_k\}$ and $\{v_k\}$ have covariances Q_k and R_k, respectively, at time t_k. The corresponding *Kalman filter equations* have the form

Time update (prediction)

$$\hat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(+),$$
$$P_k(-) = \Phi_{k-1} P_{k-1}(+) \Phi_{k-1}^T + G_{k-1} Q_{k-1} G_{k-1}^T,$$

Meas. update (correction)

$$\hat{x}_k(+) = \hat{x}_k(-) + \overline{K}_k(z_k - H_k \hat{x}_k(-)),$$
$$\overline{K}_k = P_k(-) H_k^T (H_k P_k(-) H_k^T + R)^{-1},$$
$$P_k(+) = P_k(-) - \bar{K}_k H_k P_k(-),$$

where the $(-)$ indicates the a priori values of the variables (before the information in the measurement is used) and the $(+)$ indicates the a posteriori values of the variables (after the information in the measurement is used). The variable $\overline{K}$ is the Kalman gain.

Kalman Filter vs. RLS

- RLS (from F7-1: System Identification course)

$$A(q^{-1}, \boldsymbol{\theta}) y(t) = B(q^{-1}, \boldsymbol{\theta}) u(t) + e(t)$$
$$\Rightarrow \qquad y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta} + e(t)$$

where

$$\boldsymbol{\varphi}(t) = [\, -y(t-1) \, \ldots \, -y(t-n_a) \, u(t-1) \, \ldots \, u(t-n_b)]^T$$
$$\boldsymbol{\theta} = [a_1 \, \ldots \, a_{n_a} \, b_1 \, \ldots \, b_{n_b}]^T$$

At time $t = 0$: Choose initial values of $\hat{\boldsymbol{\theta}}(0)$ and $\boldsymbol{P}(0)$

At each sampling instant, update $\boldsymbol{\varphi}(t)$ and compute

$$\hat{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t-1) + \boldsymbol{K}(t)\varepsilon(t)$$
$$\varepsilon(t) = y(t) - \boldsymbol{\varphi}^T(t)\hat{\boldsymbol{\theta}}(t-1)$$
$$\boldsymbol{K}(t) = \boldsymbol{P}(t)\boldsymbol{\varphi}(t)$$
$$\boldsymbol{P}(t) = \left[\boldsymbol{P}(t-1) - \frac{\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)}{1 + \boldsymbol{\varphi}^T(t)\boldsymbol{P}(t-1)\boldsymbol{\varphi}(t)}\right]$$

Kalman Filter vs. RLS

- KF for parameter estimation (from F7-1: System Identification)

Let us model the parameter variation according to

$$\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t) + \boldsymbol{v}(t)$$
$$y(t) = \boldsymbol{\varphi}^T(t)\boldsymbol{\theta}(t) + e(t)$$

Then

$$\hat{\boldsymbol{\theta}}(t+1) = \hat{\boldsymbol{\theta}}(t) + \boldsymbol{K}(t)\left[y(t) - \boldsymbol{\varphi}^T(t)\hat{\boldsymbol{\theta}}(t)\right]$$
$$\boldsymbol{K}(t) = \frac{\boldsymbol{P}(t)\boldsymbol{\varphi}(t)}{R_2 + \boldsymbol{\varphi}^T(t)\boldsymbol{P}(t)\boldsymbol{\varphi}(t)}$$
$$\boldsymbol{P}(t+1) = \boldsymbol{P}(t) - \frac{\boldsymbol{P}(t)\boldsymbol{\varphi}(t)\boldsymbol{\varphi}^T(t)\boldsymbol{P}(t)}{R_2 + \boldsymbol{\varphi}^T(t)\boldsymbol{P}(t)\boldsymbol{\varphi}(t)} + \boldsymbol{R}_1$$

where $\boldsymbol{v}(t)$ and $e(t)$ are independent white noise sources with $E\boldsymbol{v}(t)\boldsymbol{v}^T(t) = \boldsymbol{R}_1$ and $Ee^2(t) = R_2$.

The Continuous-Time Kalman Filter

Kalman–Bucy Filter. The *continuous-time model* for a linear stochastic system has the form

$$\frac{d}{dt}x(t) = F(t)x(t) + G(t)w(t),$$
$$z(t) = H(t)x(t) + v(t),$$

where the zero-mean uncorrelated Gaussian random processes $\{w(t)\}$ and $\{v(t)\}$ have covariances $Q(t)$ and $R(t)$, respectively, at time t. The corresponding *Kalman–Bucy filter* equations for the estimate $\hat{x}$ of the state variable x, given the output signal z, has the form

$$\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + \overline{K}(t)[z(t) - H(t)\hat{x}(t)],$$
$$\overline{K}(t) = P(t)H^{\mathrm{T}}(t)R^{-1}(t),$$
$$\frac{d}{dt}P(t) = F(t)P(t) + P(t)F^{\mathrm{T}}(t) - \overline{K}(t)R(t)\overline{K}^{\mathrm{T}}(t) + G(t)Q(t)G^{\mathrm{T}}(t).$$

Textbooks

Recommended textbooks

- [**BLK01**] Yaakov Bar-Shalom, X. Rong Li, Thiagalingam Kirubarajan, *Estimation with Applications to Tracking and Navigation*, Wiley-Interscience, June 8, 2001, ISBN: 047141655X.
- [**HM04**] David L. Hall, Sonya A. H. McMullen, *Mathematical Techniques in Multisensor Data Fusion*, Artech House Publishers; 2nd edition, March 1, 2004, ISBN: 1580533353.

Optional textbooks / references

- **[GA01]** Mohinder S. Grewal, Angus P. Andrews, *Kalman Filtering: Theory and Practice Using MATLAB*, 2nd Edition, Wiley-Interscience, January 2001, ISBN: 0-471-39254-5.
- [**HL01**] David L. Hall, James Llinas, *Handbook of Multisensor Data Fusion*, CRC Press June 20, 2001, ISBN: 0849323797.

Readings

- The discrete-time Kalman filter (DKF):
  - **BLK01**: Sections 5.1-5.3
  - **GA01**: Sections 4.1-4.2
- The continuous-time Kalman filter (CKF):
  - **BLK01**: Sections 9.1-9.2
  - **GA01**: Sections 4.3
- The extended Kalman filter:
  - **BLK01**: Sections 10.1-10.3
  - **GA01**: Sections 5.1-5.7
- Introduction to Kalman filter:
  - Welch & Bishop: Paper; Slides
  - Maybeck's book (1979)_Chapter 1