"Control Theory" - Solution #3

E5.2, E5.4, E5.5, E5.7, E5.8, P5.4

CHAPTER 5

The Performance of Feedback Control Systems

EXERCISES

E5.1 For a zero steady-state error, when the input is a step we need one integration, or a type 1 system. A type 2 system is required for $e_{ss} = 0$ for a ramp input.

E5.2 (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{100}{(s+2)(s+5)+100} \doteq \frac{100}{s^2+2\zeta\omega_n s+\omega_n^2} .$$

The steady-state error is given by

$$e_{ss} = \frac{A}{1+K_p} ,$$

where $R(s) = A/s$ and

$$K_p = \lim_{s \to 0} G(s) = \frac{100}{10} = 10 .$$

Therefore,

$$e_{ss} = \frac{A}{11} .$$

(b) The closed-loop system is a second-order system with natural frequency

$$\omega_n = \sqrt{110} ,$$

and damping ratio

$$\zeta = \frac{7}{2\sqrt{110}} = 0.334 .$$

Since the steady-state value of the output is 0.909, we must modify the percent overshoot formula which implicitly assumes that the steady-state value is 1. This requires that we scale the formula by 0.909. The percent overshoot is thus computed to be

$$P.O. = 0.909(100e^{-\pi\zeta/\sqrt{1-\zeta^2}}) = 29\% .$$

E5.3 The closed-loop transfer function is

$$\frac{Y(s)}{I(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s(s+14)+K} = \frac{K}{s^2+14s+K} .$$

Utilizing Table 5.6 in Dorf & Bishop, we find that the optimum coefficients are given by

$$s^2 + 1.4\omega_n s + \omega_n^2 .$$

We have

$$s^2 + 14s + K ,$$

so equating coefficients yields $\omega_n = 10$ and $K = \omega_n^2 = 100$. We can also compute the damping ratio as

$$\zeta = \frac{14}{2\omega_n} = 0.7 .$$

Then, using Figure 5.8 in Dorf & Bishop, we find that $P.O. \approx 5\%$.

E5.4 (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{2(s+8)}{s^2+6s+16} .$$

(b) We can expand $Y(s)$ in a partial fraction expansion as

$$Y(s) = \frac{2(s+8)}{(s^2+6s+16)}\frac{A}{s} = A\left(\frac{1}{s} - \frac{s+4}{s^2+6s+16}\right) .$$

Taking the inverse Laplace transform (using the Laplace transform tables), we find

$$y(t) = A[1 - 1.07e^{-3t}\sin(\sqrt{7}t + 1.21)] .$$

(c) Using the closed-loop transfer function, we compute $\zeta = 0.75$ and $\omega_n = 4$. Thus,

$$\frac{a}{\zeta\omega_n} = \frac{8}{3} = 2.67 ,$$

where $a = 8$. From Figure 5.13(a) in Dorf & Bishop, we find (approximately) that $P.O. = 4\%$.

(d) This is a type 1 system, thus the steady-state error is zero and $y(t) \rightarrow A$ as $t \rightarrow \infty$.

E5.5 (a) The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + GH(s)} = \frac{100}{s^2 + 100Ks + 100} ,$$

where $H(s) = 1 + Ks$ and $G(s) = 100/s^2$. The steady-state error is computed as follows:

$$e_{ss} = \lim_{s \to 0} s[R(s) - Y(s)] = \lim_{s \to 0} s[1 - T(s)]\frac{A}{s^2}$$

$$= \lim_{s \to 0} \left[1 - \frac{\frac{100}{s^2}}{1 + \frac{100}{s^2}(1 + Ks)} \right] \frac{A}{s} = KA .$$

(b) From the closed-loop transfer function, $T(s)$, we determine that $\omega_n = 10$ and

$$\zeta = \frac{100K}{2(10)} = 5K .$$

We want to choose K so that the system is critically damped, or $\zeta = 1.0$. Thus,

$$K = \frac{1}{5} = 0.20 .$$

The closed-loop system has no zeros and the poles are at

$$s_{1,2} = -50K \pm 10\sqrt{25K^2 - 1} .$$

The percent overshoot to a step input is

$$P.O. = 100e^{\frac{-5\pi K}{\sqrt{1-25K^2}}} \quad \text{for} \quad 0 < K < 0.2$$

and $P.O. = 0$ for $K > 0.2$.

E5.6 The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)} = \frac{K(s+2)}{s(s+1) + K(s+2)} = \frac{K(s+2)}{s^2 + s(K+1) + 2K} .$$

Therefore, $\omega_n = \sqrt{2K}$ and $\zeta = \frac{K+1}{2\sqrt{2K}}$. So,

$$\frac{a}{\zeta\omega_n} = \frac{4}{K+1} \, .$$

From Figure 5.13a in Dorf & Bishop, we determine that

$$\frac{a}{\zeta\omega_n} \approx 1.5$$

when $\zeta = 0.707$. Thus, solving for K yields

$$\frac{4}{K+1} = 1.5$$

or

$$K = 1.67 \, .$$

E5.7 The pole-zero map is shown in Figure E5.7. Since the dominant poles are real, you do not expect to have a large overshoot, as shown in Figure E5.7b.

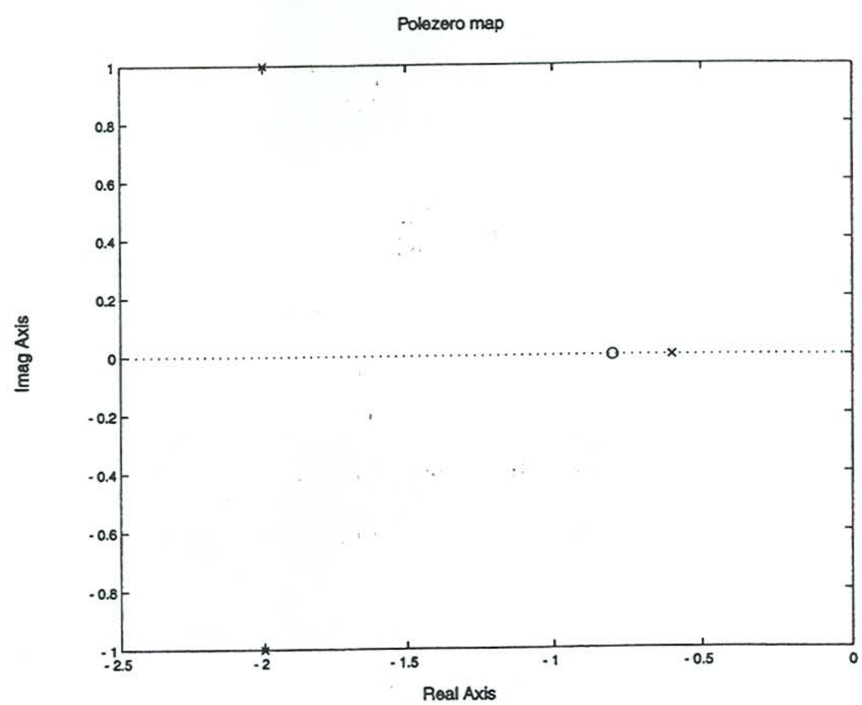

FIGURE E5.7
(a) Pole-zero map.

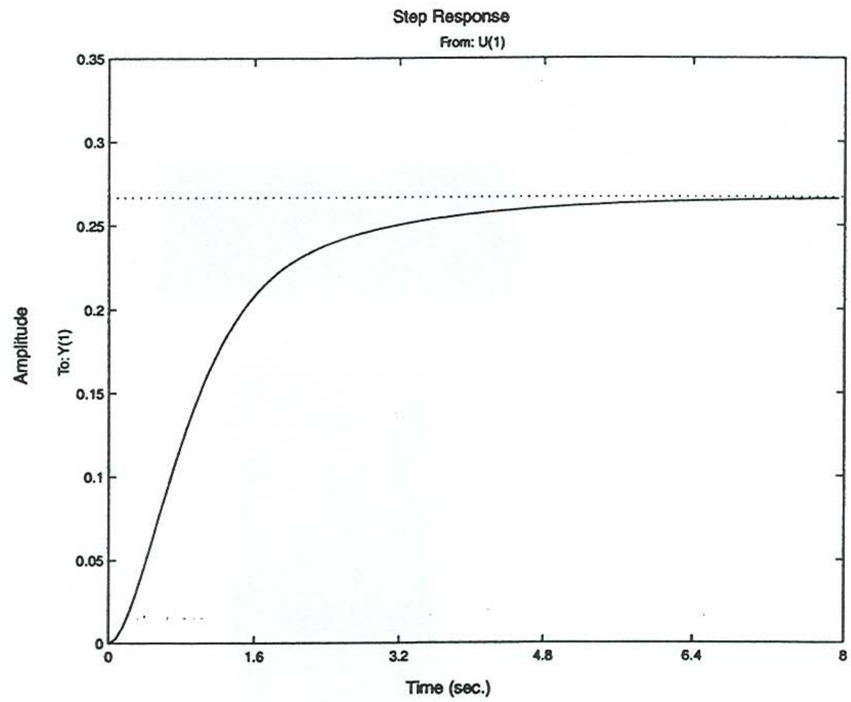

FIGURE E5.7
CONTINUED: (b) Unit step response.

E5.8 (a) The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + \sqrt{2K}s + K} .$$

The damping ratio is

$$\zeta = \frac{\sqrt{2}}{2}$$

and the natural frequency is $\omega_n = \sqrt{K}$. Therefore, we compute the percent overshoot to be

$$P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 4.3\%$$

for $\zeta = 0.707$. The settling time is estimated via

$$T_s = \frac{4}{\zeta\omega_n} = \frac{8}{\sqrt{2K}} .$$

(b) The settling time is less than 1 second whenever $K > 32$.

P5.3 Given the input

$$R(s) = \frac{1}{s^3} ,$$

we compute the steady-state error as

$$e_{ss} = \lim_{s\to 0} s \left(\frac{1}{1+G(s)} \right) \frac{1}{s^3} = \lim_{s\to 0} \left(\frac{1}{s^2 G(s)} \right) = \lim_{s\to 0} \left(\frac{1}{s^2 \left(\frac{K}{s^2}\right)} \right) = \frac{1}{K} .$$

Since we require that $e_{ss} \leq 0.5$ cm, we determine

$$K \geq 2 .$$

P5.4 (a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Thus,

$$\omega_n = \sqrt{K} \quad \text{and} \quad \zeta = 1/\omega_n = 1/\sqrt{K} .$$

Our percent overshoot requirement of 5% implies that $\zeta = 1/\sqrt{2}$, which in turn implies that $\omega_n = \sqrt{2}$. However, the corresponding time to peak would be

$$T_p = \frac{4.4}{\sqrt{2}} = 3.15 .$$

Our desired $T_p = 1.1$—we cannot meet both specification simultaneously.

(b) Let $T_p = 1.1\Delta$ and $P.O. = 0.05\Delta$, where Δ is the relaxation factor to be determined. We have that $K = \omega_n^2$ and $\zeta\omega_n = 1$, so

$$\zeta = \frac{1}{\sqrt{K}} .$$

Thus,

$$P.O. = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi/\sqrt{K-1}} .$$

Also,

$$T_p = \frac{\pi}{\sqrt{K-1}} = 1.1\Delta \, .$$

Therefore, from the proceeding two equations we determine that

$$P.O. = 0.05\Delta = e^{-1.1\Delta} \, .$$

Solving for Δ yields

$$f(\Delta) = \ln\Delta + \ln(0.05) + 1.1\Delta = 0 \, .$$

The plot of $f(\Delta)$ versus Δ is shown in Figure P5.4. From the plot we see

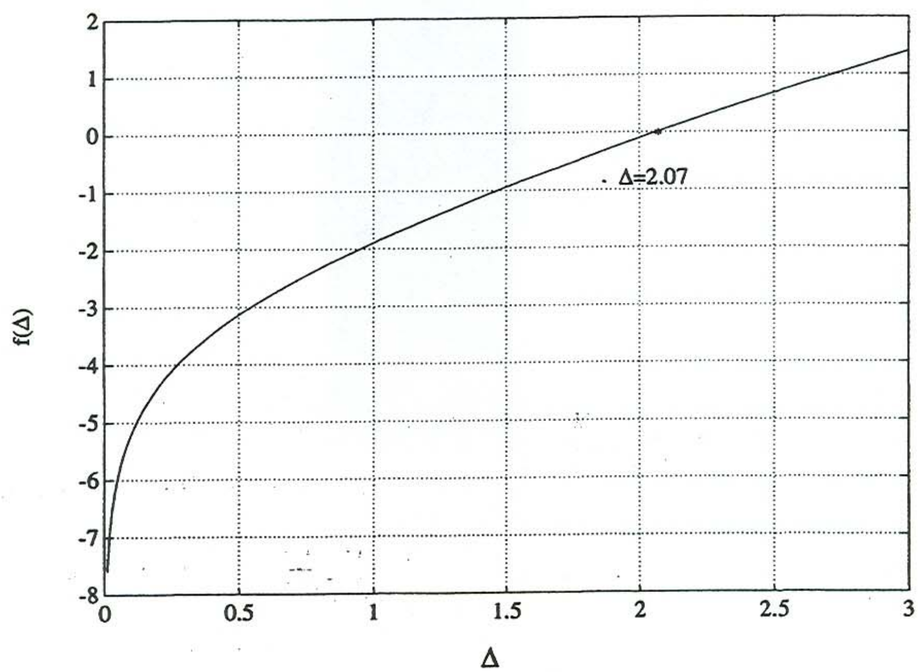

FIGURE P5.4
Solving for the zeros of *f*.

that $\Delta = 2.07$ results in $f(\Delta) = 0$. Thus,

$$P.O. = 0.05\Delta = 10\%$$
$$T_p = 1.1\Delta = 2.3 \text{ sec.}$$

So, we can meet the specifications if they are relaxed by a factor of about 2 (i.e. $\Delta = 2.07$).