Multilayered linear dispersion codes for flat fading multiple-input multiple-output channels

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Abstract: The authors investigate the design of linear dispersion (LD) codes, aiming at flexible encoding schemes that allow various rate–performance tradeoffs under a common coding structure. First, the capacity of LD codes is studied. It is shown that the maximum attainable multiplexing gain of a linear dispersion code is the number of symbols per channel use of the code (i.e. coding rate in symbols). In addition, conditions on the construction of linear dispersion matrices for various multiplexing gains are established. A general multilayered linear dispersion coding scheme that allows various multiplexing gains is then proposed. In the proposed scheme, coding rate can be adapted by employing different numbers of dispersion matrices. Furthermore, phase shifting among input symbols is applied to optimise the error performance without loss of multiplexing gain. The construction of dispersion matrices and the optimisation of the phase shifts together constitute a structured approach for the design of linear dispersion codes. Simulation results demonstrate that the new codes outperform conventional LD codes at various data rates.

1 Introduction

A remarkable recent advance in wireless communications is the so-called multiple-input multiple-output (MIMO) technology. It makes use of multiple transmit and receive antennas to achieve higher data rate and reliability over fading channels [1, 2]. To date, numerous MIMO transmission schemes have been developed. Early examples include space–time trellis codes (STTC) [3] and space–time block codes (STBC) [4, 5] aimed at improved error performance (reliability) and spatial multiplexing schemes [6, 7] aimed at higher data rate. A common drawback of early MIMO transmission schemes is that they are not flexible in rate–performance tradeoff.

Interestingly, Zheng and Tse have shown that there exists an optimal tradeoff curve between diversity and multiplexing [8]. Since the diversity gain characterises the error performance as a function of signal-to-noise ratio (SNR) and multiplexing gain determines the data rate, the multiplexing–diversity tradeoff is a tradeoff between rate and performance. Although it was shown in [8] that diagonal Bell Labs Layered Space–Time Architecture (D-BLAST) [6] allows optimal multiplexing–diversity tradeoff, it is not clear how to achieve this with structured codes instead of Gaussian random codes. In addition, space–time codes carved from properly constructed lattices achieve optimal multiplexing–diversity tradeoff for flat fading channels [9]. However, the decoding complexity is still prohibitively high even for a moderate number of transmit and receive antennas.

Recently, the need to support multirate wireless services has attracted a great deal of research attention to the design of MIMO transmission schemes that provide flexible tradeoff between rate and performance. Among these, LD coding [10–15] has notable advantages. It includes vertical Bell Labs layered space–time architecture (V-BLAST) [7] and many existing space–time block codes [5] as special cases, and allows suboptimal linear receivers with greatly reduced complexity [10]. There also exist scalable LD codes [13–15] that allow an easy rate adaptation and, hence, flexible multiplexing–diversity tradeoff. The advantages of the LD codes, however, come at the cost of difficulties in design. This is primarily because a simultaneous optimisation of error performance and rate is highly nonlinear. In [10], only the mutual information between transmitted and received signals is maximised.
This approach does not consider diversity and coding gains. As a result, performance may not be satisfactory. In [11], diversity gain is included in the optimisation objectives in addition to the mutual information. The resulting designs only yield limited performance improvement unless SNR is very high. This is because the diversity gain as the only performance index is misleading, owing to the possible existence of very small eigenvalues and hence small determinants of the difference codeword matrices [12]. To remedy this problem, coding gain was also taken into account in [12] but at the cost of loss in channel capacity. The problem can be made simpler in structured approaches [13–15]. However, the design remains difficult. There also exist designs based on number theory but only for very limited applications [16–18].

To allow flexible and efficient multiplexing–diversity tradeoff, it is desirable to have a combined spatial multiplexing and coding approach. In this paper, the design of LD codes is investigated, aiming at flexible encoding schemes that allow various rate–performance tradeoffs. On the basis of the mutual information between the transmitted and the received signals, conditions for linear dispersion matrices to achieve various multiplexing gains are established. The results allow us to develop a general multilayered linear dispersion coding scheme that provides various multiplexing gains simply by varying the number of dispersion matrices. Furthermore, without the loss of mutual information, both diversity and coding gains are maximised by applying phase shifting among input symbols. Because of the regular structure of the proposed coding scheme, the optimisation of phase shifts is simple and can be efficiently carried out by an off-line computer search. This substantially simplifies the selection of symbol constellation and the number of layers (multiplexing gain) in multirate applications.

The rest of the paper is organised as follows. In Section 2, the system model is described and a formal approach for the design of linear dispersion codes is developed. On the basis of this formal approach, a general multilayered linear dispersion coding scheme that allows various rate–performance tradeoffs is proposed in Section 3. In Section 4, we compare the proposed multilayered codes with some existing LD codes. Simulation results are presented in Section 5 and conclusions are drawn in Section 6.

2 Design of linear dispersion codes

In this section, we present the system model and describe our approach to the design of LD space–time codes.

We consider an MIMO communication system consisting of \( N_t \) transmit and \( N_r \) receive antennas over a flat fading channel. The complex gain of the channel between transmit antenna \( n \) and receive antenna \( m \) is denoted by \( h_{mn} \). It is assumed that the channel gains are samples of independent identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with zero mean and variance 1 [i.e. \( h_{mn} \sim CN(0, 1) \)], and are perfectly known to the receiver but unknown to the transmitter. The channel is assumed to undergo block fading, that is, it keeps constant over the duration of a block and varies independently from block to block [3, 5, 19–20].

Linear dispersion codes are space–time block codes. With a block of \( K \) input symbols, \( s = [s_1, s_2, \ldots, s_K]^T \), an \( N_t \times T \) codeword matrix is constructed as in [10]

\[
S = \sum_{k=1}^{K} M_{sk} + \sum_{k=1}^{K} N_{sk}^* \tag{1}
\]

where \( M_s, N_s \in \mathbb{C}^{N_t \times T} \) are the dispersion matrices for the \( k \)th symbol. Since the use of dispersion matrices for the conjugates of input symbols does not provide evident improvement [10], only the dispersion matrices for the original input symbols will be considered in this study, that is

\[
S = \sum_{k=1}^{K} M_{sk} \tag{2}
\]

Within one block, the received signal can be written as

\[
Y = \sqrt{P/N_t}HS + V \tag{3}
\]

where \( H \in \mathbb{C}^{N_t \times N_r} \) is the channel matrix whose \( (m, n) \)th entry is \( h_{mn} \), \( V \in \mathbb{C}^{N_t \times T} \) the additive complex Gaussian noise matrix with i.i.d. entries, that is, \( v_{mn} \sim CN(0, N_0) \) and \( P \) the total transmitted power.

Let \( y \) be the vector of length \( N_rT \) formed by stacking the \( T \) columns of \( Y \), that is, \( y = \text{vec}(Y) \), and similarly let \( v = \text{vec}(V) \). Then (3) can be rewritten as

\[
y = \sqrt{P/N_t}HS + v = \sqrt{P/N_t}Hs + v \tag{4}
\]

where \( M = [\text{vec}(M_1), \text{vec}(M_2), \ldots, \text{vec}(M_K)] \) with \( \text{tr}(M^H M) = N_rT \), and \( H = I_T \otimes H \) with \( \otimes \) as the kronecker product operator.

Evidently, from (2), a linear dispersion code is completely characterised by its \( K \) dispersion matrices, \( \{M_k\}_{k=1}^{K} \). Below, we will refer to a linear dispersion code with \( K \) inputs and output matrix of size \( N_r \times T \) as a \((T, K)\) LD code. If the input symbols are drawn from a finite alphabet of size \( Q \), the coding rate of a \((T, K)\) LD code is \( K \log_2 Q/T \) bits per channel use. Another useful definition is the coding rate in symbols, or simply symbol rate, which is \( R = K/T \) for a \((T, K)\) LD code. As can be seen, there exist salient design
choices for achieving different coding rate. A well-designed LD encoding scheme shall support a variety of multiplexing–diversity tradeoffs by varying the constellation size \( Q \) and/or the symbol rate \( R \). This makes LD codes particularly attractive for future multirate wireless communications.

The performance of an LD code depends not only on the set of dispersion matrices but also on the alphabet of the input symbols. This with the large number of possible codewords even for a reasonable value of \( K \) makes the design of an LD code difficult. In particular, a direct application of Tarokh’s well-known criteria to the design of LD codes is impractical and also inappropriate [10]. Instead, maximising the mutual information has been used as the primary criterion in the design of LD codes [10]. In this design method, Gaussian input was often assumed in order to make the optimisation problem tractable. The resulting LD codes do not necessarily lead to optimal performance for symbols drawn from finite alphabets. To alleviate this problem, error probability measures have been used in some existing designs. A typical error performance measure is the diversity gain [11]. However, diversity gain can be misleading. Small eigenvalues in the difference matrix of a pair of codewords may not contribute to the error exponent unless the SNR is very large. Other measures including coding gain [12], union error exponent unless the SNR is very large. Other measures including coding gain [12], union error exponent unless the SNR is very large. Other measures including coding gain [12], union

\[
E[I(x;y)] = E \log_2 \left( I_{N,1} + \frac{P}{N_1 N_0} HH^H \right)
\]

\[
\leq E \log_2 \left( I_{N,1} + \frac{P}{N_1 N_0} HD H^H \right)
\]

\[
= \sum_{i=1}^{T} E \log_2 \det \left( I_{N_i} + \frac{P}{N_i N_0} HD H^H \right)
\]

The equality in (6) holds when all other entries in \( MM_H \) that are excluded in \( D_i \) for \( i = 1, 2, \ldots, T \) are zeros, that is

\[
G_i G_i^H = 0_{N_i \times N_i}, \quad \forall i \neq j
\]

We now consider the optimisation of each term in the summation of the last equation of (6). First, note that the maximum rank of \( MM_H \) is \( RT \), hence we assume that rank \( (D_i) = R \). Let trace \( (D_i) = C_i \). Apparently, \( \sum_{i=1}^{T} C_i = \text{trace}(MM_H) = N_r T \). Since \( D_i \) is symmetric and positive semi-definite, one can write \( D_i = U_i A_i U_i^H \) with \( A_i \) being diagonal and \( U_i \) unitary. Furthermore, \( A_i \) has \( R \) non-zero entries, say they are the \( i_1 \text{th}, i_2 \text{th}, \ldots, i_R \text{th} \) diagonal entries. Let \( \tilde{A}_i \) be the \( R \times R \) matrix obtained by collecting the rows and columns \( i_1, i_2, \ldots, i_R \) of \( A_i \). Similarly, let \( \tilde{U}_i \) be the \( N_i \times R \) matrix that collects the columns \( i_1, i_2, \ldots, i_R \) of \( U_i \). Then, it is clear that

\[
E \log_2 \det \left( I_{N_i} + \frac{P}{N_i N_0} HH^H \right)
\]

\[
= E \log_2 \det \left( I_{N_i} + \frac{P}{N_i N_0} \tilde{H} \tilde{A}_i \tilde{H}_i^H \right)
\]

where \( \tilde{H} = H \tilde{U} \). It can be easily checked that the entries of \( \tilde{H} \) are i.i.d. complex Gaussian random variables with zero mean and variance 1. Hence, (8) is maximised when the diagonal entries in \( \tilde{A}_i \) are equal and \( A_i = (C_i/R)I_R \) [11]. Then (6) can be rewritten as

\[
E[I(x;y)] \leq \sum_{i=1}^{T} E \log_2 \det \left( I_{N_i} + \frac{PC_i}{N_i N_0 R} \tilde{H} \tilde{H}_i^H \right)
\]

Note that \( \log_2 \det (I_{N_i} + x\tilde{H} \tilde{H}_i^H) \) is a concave function of \( x \) and so is \( E \log_2 \det (I_{N_i} + x\tilde{H} \tilde{H}_i^H) \). Further, by the fact that \( \sum_{i=1}^{T} C_i = N_r T \), (9) is maximised when
$G_i = N_i$, for $i = 1, 2, \ldots, T$; that is, $G_iG_i^H = D$ where $D$ is only diagonal matrix of $R$ non-zero entries of value $N_i/R$. This with (7), has proved Theorem 1.

Two corollaries follow from Theorem 1.

**Corollary 1:** The maximum achievable multiplexing gain of an LD code of symbol rate $R \leq N_t$ is min$(R, N_t)$.

A coding scheme is said to preserve the channel capacity if for any realization of the channel, say $H$, the instantaneous capacity of the coded channel (i.e. the cascade of the encoder and the physical channel) is

$$\log_2 \det \left( I_i + \frac{P}{N_i N_0} HH^H \right)$$

**Corollary 2:** With independent inputs, a linear dispersion code preserves the channel capacity if and only if the dispersion matrices satisfy

$$\text{tr}(M_i^H M_j) = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{otherwise} \end{cases}$$

**Proof:** Let $R = N_t$ in (5). Hence, from Theorem 1, the mutual information is maximised when $D = I_{N_t}$ and $MM^H = I_{N_t}$. As such, the instantaneous channel capacity is

$$I(x; y) = \log_2 \det \left( I_{N_t} + \frac{P}{N_i N_0} HH^H \right)$$

This has shown the sufficiency of the conditions. To establish the necessity, assume $R < N_t$ or (10) does not hold, then it is easy to show that $I(x; y)$ is smaller than that given by (11). Hence, the corollary has been proved.

Note that Corollary 2 was also proved in [23]. We are now ready to make some observations regarding to the construction of linear dispersion matrices.

1. Recall that the $n$th row of $G_i$ collects the dispersion weights of all symbols at transmit antenna $n$ and time $t$, that is, $G_i(n, t) = [M_1(n, t), M_2(n, t), \ldots, M_{K}(n, t)]$ with $M_k(n, t)$ being the $(n, t)$th entry of $M_k$. Hence we can call the $n$th row of $G_i$ as the dispersion vector associated with $n$ and time $t$. Equation (5) specifies that all the dispersion vectors are either zero or mutually orthogonal to each other. If one choose a multiplexing gain $R \leq N_t$, then there are exactly $R$ non-zero dispersion vectors at a given time. In other words, $R$ transmit antennas are occupied at any time.

2. If $M_i$ and $M_j$ have non-zero entries at different locations, that is, $M_i(n, t)M_j(n, t) = 0, \forall (n, t)$, then $\text{tr}(M_i^H M_j) = 0$.

3. For full diversity, a necessary condition is that each dispersion matrix must have at least $N_t$ non-zero entries at different rows and different columns.

Our objective is to design a set of $N_t T$ dispersion matrices for a coding block of size $T$. This set of dispersion matrices must satisfy (10) to ensure full channel capacity and multiplexing gain $R = N_t$. If the multiplexing gain $R$ is chosen to be less than $N_t$, one can simply take $RT$ dispersion matrices from the set that satisfy (5). With this objective in mind and on the basis of the above observations, we may construct the dispersion matrices as follows.

**Rule 1:** The $N_t T$ dispersion matrices are divided into $N_t$ groups, each of which has $T$ matrices. Any two matrices from different groups must have non-zero entries at different locations. This guarantees that the trace of the Hermitian product of any two matrices from different groups be zero, that is, satisfy (10).

**Rule 2:** Assign each dispersion matrix $N_t$ non-zero entries at different columns and rows. This ensures that a) full diversity is possible, b) any two dispersion vectors associated either with the same transmit antenna or the same transmit time are orthogonal to each other.

**Rule 3:** Within a group, apply spatial weighting to make sure any two dispersion vectors associated with different times and different antennas are orthogonal, that is satisfy (5). This can be readily done by applying spatial weightings as will be shown in the next section.

### 3 Proposed scheme

In this section, the design rules established in the last section will be used to develop a new multilayered linear dispersion coding scheme.

In the proposed scheme the encoding is performed block by block. Each encoding block takes $RT$ symbols as input, (i.e. $K = RT$), $T$ symbols per layer (group) and output a codeword matrix of size $N_t \times T$, where $T$ is an integer multiple of $N_t$. Hence, the symbol rate is $R$ if $R$ layers are used. The dispersion matrix for the $t$th symbol of layer $i$ is given by

$$M_{t,i} = e^{i\phi_1} e^{i\theta} W_i P_i A_{(t-1)/N_t}^\dagger \sqrt{N} \quad t = 1, \ldots, T, \quad i = 1, \ldots, R$$

where $\{\phi_1\}_{i=1}^{N_t}$ is a set of phase shifts among symbols in each layer, $\{\theta\}_{i=1}^{R}$ is a set of phase shifts among layers, $W_i$ is an $N_t \times N_t$ diagonal matrix whose $n$th diagonal entry is $\exp\left(-i2\pi(n-1)(t-1)/N_t\right)$, $P_i$ is an $N_t \times T$ circulant matrix whose first row has 1 as its $i$th entry and zeros...
elsewhere, that is
\[
P_i = \begin{bmatrix}
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]
(13)

\[A\] is a \(T \times T\) circulant matrix given by
\[
A = \begin{bmatrix}
N & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]
(14)

where \([x]\) is the greatest integer smaller than or equal to \(x\).

It can be readily verified that the dispersion matrices specified in (12) satisfy (5) and (10) regardless of the values of phase shifts \(\phi_t\) and \(\theta_t\). Therefore the choice of \(\phi_t\) and \(\theta_t\) will not affect the attainable mutual information. By Corollary 1, for a given number of layers \(1 \leq R \leq N_t\), a multiplexing gain of \(\min(R, N_t)\) is possible. In addition, by Corollary 2, if \(R = N_t\) layers are used, the proposed construction in (12) preserves channel capacity.

It is interesting to note that the proposed scheme is a generalisation of the scheme proposed in [11]. The proposed scheme here is developed rigorously based on Theorem 1 and applies to any block length of \(T\). Following this development, it is now clear that the number of layers directly determines the multiplexing gain and, hence, data rate. Furthermore, one can select the phase shifts in [11] to ensure full rank of the difference matrix between any two codeword pairs. The resultant codes may not have the smallest Euclidean distances. As will be shown shortly, the average Euclidean distance of a pair of codewords \(S\) and \(S'\) can be written as
\[
d(S, S') = E[\text{tr}(H(S - S')S - S')^H H^H]
\]
(18)

As can be seen from (17) and (18), a diagonal of the codeword matrix \(S\) only depends on the \(N_t\) symbols within one layer. Consequently, codeword pairs differing at only one diagonal are most error prone in that they have the smallest Euclidean distances. As will be shown shortly, the coding gain and diversity gain of these error-prone codeword pairs are independent of phase shifts \(\theta_t\). This suggests an optimisation procedure with two steps:

Step 1: Choose \(\{\phi_t\}_{t=1}^{N_t}\) to maximize the diversity and coding gains of codeword pairs that differ at only one diagonal.

Step 2: For the given phase shifts \(\{\phi_t\}_{t=1}^{N_t}\) from Step 1, find \(\{\theta_t\}_{t=1}^{N_t}\) to maximise the minimum diversity gain and coding gain among all codeword pairs.

To find the optimum set of phase shifts \(\{\phi_t\}_{t=1}^{N_t}\), we consider two codewords \(S\) and \(S'\) that differ only at one diagonal, say diagonal 1. The square root of the determinant of \((S - S')(S - S')^H\) is the product of \(N_t\) distances given by
\[
D = \prod_{l=1}^{N_t} d_l
\]
(19)
where

\[
ed = \frac{e^{i\theta}q_{11}(s_{1,1} - s'_{1,1}) + \ldots + e^{i\theta_s}q_{N_s}(s_{N_s,1} - s'_{N_s,1})}{\sqrt{R}}
\]

(20)

If \( D \) is non-zero, the codeword pair has diversity gain \( N_tN_r \) and the associated coding gain is

\[
G_c = \frac{D^{2/N_t}}{N_t}
\]

It is clear that (19) is independent of phase shifts because \( |e^{i\theta}| = 1 \) for any value of \( \theta \). Note that \( \theta = \theta_{t,s,N_t} \). To optimise the performance of those vulnerable codeword pairs, and hence the associated diversity and coding gains, the phase shifts among symbols \( \{\theta_{t,s,N_t}\} \) are chosen to maximize the minimum determinant \( D \) taken over all possible pairs \( \hat{s} = (s_{1,1}, s_{2,2}, \ldots, s_{N_s,1}) \) and \( \hat{s}' = (s'_{1,1}, s'_{2,2}, \ldots, s'_{N_s,1}) \) in one layer, that is

\[
\theta_t = \arg \max_{\{\theta_t\}} \min_{[\hat{s}]_t} D, \quad t = 1, 2, \ldots, N_t \quad (21)
\]

Since the symbols are taken from finite alphabets, the optimal \( D \) must be positive. In fact, when the constellation is a complex integer ring, one can readily find phase shifts to ensure a positive \( D \) [11]. If only one layer is used, the phase shifts obtained according to (21) maximize the diversity and coding gains of the code. That is, they maximize the diversity and coding gains of the codeword pairs that have small Euclidean distance. In addition, the diversity gain of the code is bounded as stated in the following theorem.

**Theorem 2:** In the proposed scheme, for a given symbol rate \( 1 \leq R \leq N_t \), regardless of the values of \( \{\theta_{t,s,N_t}\} \) and the choice of symbol constellation, the diversity gain is bounded as

\[
G_d \geq (N_t - R + 1)N_t
\]

(22)

**Proof:** First, we note the following facts:

(a) With phase shifts \( \{\theta_{t,s,N_t}\} \) as defined in (21), if there is a non-zero entry in a diagonal of a difference matrix \( E = S - S' \), then all the entries at that diagonal must be non-zero.

(b) Within any contiguous \( N_t \) diagonals of a difference matrix \( E \), there are at most \( R \) contiguous non-zero diagonals and at least \( N_t - R \) contiguous zero diagonals.

By the above two facts, in any difference matrix \( E \), one can always find a square upper triangular submatrix of size \( N_t - R + 1 \) with non-zero diagonal entries. The rank of this submatrix is full. Hence, the rank of \( E \) is at least \( N_t - R + 1 \). Then the diversity gain \( G_d > (N_t - R + 1)N_t \).

It should be mentioned that for constellations drawn from finite alphabet, full rank \( N_tN_r \) is always possible for the difference matrices of any pair of distinct codeword matrices [11]. However, with practical SNR, some of the eigenvalues could be very small, leading to a smaller decay rate of error performance. Note that the maximum multiplexing gain of the MIMO channel is \( \min(N_t, N_r) \). Hence, we only consider the number of layers \( R \leq \min(N_t, N_r) \). In such a case, the above theorem shows that the diversity gain of the proposed scheme is usually large except when \( N_t \) and \( N_r \) are small and \( R \) is close to \( \min(N_t, N_r) \). It is worthy to note that the improvement in performance quickly diminishes as the diversity gain increases beyond 4 [3], and in such a case, the Euclidean distance can be a good performance measure [24]. Furthermore, following the same procedure as in the proof of Theorem 2, one can show that a pair of codewords with diversity gain \( (N_t - g)N_t \) must differ at least \( g + 1 \) diagonals. Consequently, from (18) their Euclidean distance must be at least \( g + 1 \) times of the minimum Euclidean distance of the code. All these suggest that, when \( N_t \) and \( N_r \) are large, the first optimisation step ensures nearly optimum performance and the second step can be omitted, that is, no phase shifts among layers are required.

When \( N_t \) is small and \( R \) is close \( N_t \), the effective diversity gain is important. In such a case, one can apply phase shifts \( \{\theta_{t,s,N_t}\} \) to ensure full diversity such as that in [11]. However, the coding gain can be close to zero. Therefore we apply phase shifts \( \{\theta_{t,s,N_t}\} \) among layers to maximise diversity and coding gains for all the codeword pairs. This can be done by maximising the minimum coding gain \( G_c \) over all possible codeword pairs, that is

\[
\theta_t = \arg \max_{\{\theta_t\}} \min_{[\hat{s}, \hat{s}']} G_{c, i}, \quad i = 1, 2, \ldots, R
\]

(23)

Equations (21) and (23) can be solved easily and efficiently by using an off-line computer search unless the constellation is very large.

In Tables 1–3, we show some results obtained by applying the proposed phase shifting scheme. In these tables, it is assumed that \( N_t = 4 \) and \( \phi_s = \theta_t = 1 \). Because the constellations are drawn from finite alphabets, all the codes in the tables have full diversity gain. Hence, only the coding gains are provided in the tables. As shown in Tables 1–3, with the same transmitted power and data rate the multilayered codes provide larger coding gains than the single-layered codes. For example, the two-layered code with QPSK modulation has a much larger coding gain
than the one-layered codes with 16PSK or 16QAM modulation. When three transmit antennas are used in Table 3, the three-layered code with BPSK modulation has a much larger coding gain than the one-layer code with 8PSK modulation. These show that one shall always try to use a larger number of layers provided it is no larger than $\min(N_r, N_t)$. For large diversity gains such as the last two codes in Table 3, only phase shifts within layers $\{\phi_i\}_{i=1}^2$ are applied.

4 Comparison of linear dispersion codes

In this section, we compare the proposed multilayered codes with some existing LD codes.

4.1 Alamouti’s scheme

Consider channels with two transmit antennas, for which the Alamouti’s scheme is designed. If the number of receive antennas is more than 1, then a multiplexing gain of 2 is possible and two layers can be used in our scheme. Suppose $T=2$ and denote the four input symbols by $s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}$. If the input symbols are taken from a real constellation (e.g. PAM), then the optimal phase shifts will be $\phi_0 = 0$, $\phi_2 = \pi/2$, $\theta_1 = 0$, $\theta_2 = \pi/2$, and the transmitted code word is

$$ s = \begin{bmatrix} s_{1,1} + js_{2,1} \\ js_{1,2} + js_{2,2} \\ js_{1,2} - js_{2,2} \\ s_{1,1} - js_{2,1} \end{bmatrix} $$

(24)

Table 2 Optimum phase shifts and corresponding coding gains for two-layered codes over two transmitted antennas

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$N_t$</th>
<th>$R$</th>
<th>$\phi_2$, $^\circ$</th>
<th>$\theta_2$, $^\circ$</th>
<th>$G_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>2</td>
<td>2</td>
<td>90</td>
<td>90</td>
<td>2</td>
</tr>
<tr>
<td>QPSK</td>
<td>2</td>
<td>2</td>
<td>45</td>
<td>30, 60, 120, 150</td>
<td>0.2588</td>
</tr>
<tr>
<td>8PSK</td>
<td>2</td>
<td>2</td>
<td>22.5</td>
<td>13, 32, 58, 77</td>
<td>0.0312</td>
</tr>
<tr>
<td>8QAM</td>
<td>2</td>
<td>2</td>
<td>33.5</td>
<td>38</td>
<td>0.0534</td>
</tr>
<tr>
<td>16QAM</td>
<td>2</td>
<td>2</td>
<td>45</td>
<td>18.5, 26.5, 63.5, 71.5</td>
<td>0.0416</td>
</tr>
</tbody>
</table>

Table 3 Optimum phase shifts and corresponding coding gains for codes over three transmitted antennas

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$N_t$</th>
<th>$R$</th>
<th>$\phi_2$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$G_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>3</td>
<td>3</td>
<td>128</td>
<td>20</td>
<td>160</td>
<td>0.1255</td>
</tr>
<tr>
<td>8PSK</td>
<td>3</td>
<td>1</td>
<td>63</td>
<td>N/</td>
<td>N/</td>
<td>0.0438</td>
</tr>
<tr>
<td>8QAM</td>
<td>3</td>
<td>1</td>
<td>22,</td>
<td>112</td>
<td>N/</td>
<td>0.1069</td>
</tr>
</tbody>
</table>

Define $x_1 = s_{1,1} + js_{2,1}$ and $x_2 = js_{1,2} - js_{2,2}$.

$$ s = \begin{bmatrix} x_1 \\ x_2 \\ -\bar{x}_2 \\ \bar{x}_1 \end{bmatrix} $$

(25)

which is exactly the Alamouti’s scheme. Thus, the proposed scheme subsumes the Alamouti’s scheme as a special case. For instance, the Alamouti’s scheme with 16QAM is identical of the proposed two-layered scheme with 4PAM and $N_t = T = 2$. Although it is a two-layered scheme, the symbols are limited to be real. As such, it loses half the channel capacity at high SNR when the number of receive antennas is two. On the other hand, the proposed scheme is possible to achieve full channel capacity regardless the number of receive antennas.

4.2 Damen’s scheme

Damen et al. proposed LD codes for systems with two transmit antennas based on number theory [14]. The codeword is constructed as

$$ s = \begin{bmatrix} s_{1,1} + e^{i\theta}s_{2,1} \\ e^{i\theta}(s_{1,2} - e^{i\theta}s_{2,2}) \\ s_{1,1} - e^{i\theta}s_{2,1} \end{bmatrix} $$

(26)

with $\theta = \phi/2$. Comparing (26) with (12) when $R = N_t = T = 2$, the two schemes have identical construction except that $\theta$ is constrained to be $\phi/2$ in Damen’s scheme. This constraint simplifies the search of phase shifts but leads to loss in coding gain. On the other hand, in the proposed scheme, $\phi$ is chosen to maximise the coding gain of the codeword pairs that differ only in one layer instead of the coding gain of the code. As such, the complexity of the search of $\phi$ is reduced from $O(g^N)$ to $O(g)$ where $g$ is a function of the constellation and $R$ is the number of layers. Although $\theta$ has also to be searched in the proposed scheme, the search is only needed when the constellation size is small. For instance, for QPSK constellation, Damen’s scheme yields the optimal coding gain 0.2369 at $\phi = 2\theta = 0.5$, whereas the proposed scheme yields the optimal coding gain 0.2588 at $\phi = \pi/4$ and $\theta = \pi/6$. Note that Damen’s scheme only applies for channels with two transmit antennas.
4.3 Ma’s full-diversity full-rate codes

As mentioned before, the proposed scheme can be seen as a generalization of Ma’s scheme [11], although developed with different approaches. One of the main differences is the optimization of phase shifts. In Ma’s scheme, the phase shifts are chosen only to ensure full rank of the difference matrix without optimizing the coding gain. This can lead to solutions with unsatisfactory performance because the existence of a large portion of codeword pairs with small effective diversity gains [10]. While a direct optimization of the error performance is intractable, a better strategy, which is taken in this paper, is to ensure large coding gain and diversity gain of the codeword pairs that have small Euclidean distance. Further performance comparison of the two schemes will be presented in the next section.

5 Simulation results

In this section, we provide simulation results to compare the proposed scheme with several existing LD codes. The channel model described in section 2 was assumed and maximum likelihood decoding was performed for all the schemes. Hence, for a given data rate, all the schemes shown here except Alamouti’s code have the same decoding complexity. Since all the codes here are LD codes, suboptimal decoders can be used.

Fig. 1 compares the proposed scheme with Hassibi’s scheme [10] and Ma’s scheme [11] over a channel with two transmit and two receive antennas. The code with the best performance in [10] was chosen. In all the schemes, two layers are employed with BPSK modulation. The modulation block is chosen to be four channel uses, that is, 8 bits per block. As can be observed from the figure, at block error rate (BLER) $= 2 \times 10^{-4}$, a performance gain of approximately 3 dB is achieved for the proposed scheme over Hassibi’s scheme and 1.8 dB over Ma’s scheme. This is because the proposed scheme maximises the diversity and coding gains without loss of mutual information. Hassibi’s code preserves channel capacity but does not guarantee good performance since the diversity and coding gains were not explicitly optimised. Ma’s code maximises the diversity gain and preserves channel capacity but does not necessarily yield high coding gain which affects the performance as discussed before.

In Fig. 2, the performance curves of several two-layered codes with QPSK constellation are provided. Again, two transmit and two receive antennas are assumed. As expected, similar behaviour as in the case of Fig. 1 is observed. Specifically, the proposed two-layered code performs the best, followed by Damen’s, Ma’s and Hassibi’s codes. In addition, performance curves of Alamouti’s code and the proposed one-layered code, both with 16QAM, are provided. As mentioned before, Alamouti’s code with 16QAM is identical with the proposed two-layered code with 4PAM. Hence, its performance is the worst among the two-layered codes. To examine the effects of the number of layers or symbol coding rate on performance, one can compare the proposed two- and one-layered codes. As can be seen, with the same data rate, the proposed two-layered code has approximately 2.5 dB gain over the proposed one-layered code at BLER = 0.01. From Tables 1 and 2, the coding gain of the one-layered code with 16QAM is 0.2 and that of the two-layered code with QPSK is 0.2588. The actual performance gap is larger than indicated by the coding gain at BLER = 0.01. However, the performance gap is expected to reduce at higher SNR. Nevertheless, the two-layered code will always outperform. This shows that, for a given data rate, one shall choose a larger number of layers.
against constellation size to achieve better performance as predicted by Theorem 1.

Last, we examine the effects of the number of layers on a channel that consists of multiple transmit antennas but only one receive antenna. In such a case, the available multiplexing gain is 1. However, codes with more layers (i.e., higher coding rate in symbols) are expected to perform better in a practical SNR range. Specifically, three transmit antennas were assumed. Performance curves of three full-rate codes: the proposed, Hassibi’s and Ma’s codes, are illustrated in Fig. 3. With BPSK and $T = 3$, each modulation block consists of 9 bits. In addition, the performance of the proposed one-layered code with 8PSK is also provided. As can be seen, the proposed three-layered code performs significantly better than others. At BLER $= 2 \times 10^{-3}$, the proposed three-layered code has a performance gain about 2 dB over the other two full-rate codes. It is interesting to note in this case, the proposed one-layered code outperforms the other two full-rate codes when SNR is $>20$ dB. This is because the proposed one-layered code ensures full-diversity gain and optimal coding gain. Although Ma’s code also enjoys full diversity gain, some of its diversity orders may not contribute to the decay of error performance till SNR is extremely high. It is also interesting to note that the two curves of the proposed 3- and one-layered codes consistently have the same slopes, with the former outperforming the latter. This, again, demonstrates that one shall maximise the number of layers for a given target data rate. However, the effects of the number of layers is less significant than the case shown in Fig. 2 because there is only one receive antenna.

In summary, the above results show that the proposed scheme provides flexible rate–performance tradeoff and significant performance gain when compared with the conventional schemes.

6 Conclusions

In this paper, we have investigated the design of LD codes with flexible rate–performance tradeoff. It shows that the attainable multiplexing gain of a linear dispersion code is exactly the symbol rate defined as the number of symbols per channel use of the code. This also implies that full-rate LD codes have to be used to preserve the channel capacity. Conditions on the dispersion matrices for various multiplexing gains were also established. With these results, a general multilayered LD coding scheme has been proposed. With the new coding scheme, various multiplexing gains can be obtained simply by increasing the coding rate and augmenting the set of existing dispersion matrices. Furthermore, the error performance is optimised by applying phase shifts among input symbols without the loss of mutual information. Simulation results demonstrated that the proposed coding scheme significantly outperforms existing space–time block codes under various data rates.

7 References


