INSE6320	Jia Yuan Yu
2: Examples	
Concordia	January 14, 2020

Engineering examples.

1 Stochastic Dominance

In the assignment, we have compared the costs S and Σ associated with two decisions. What is another method for comparing two random variables X and Y (or their probability distribution functions F_X and F_Y)?

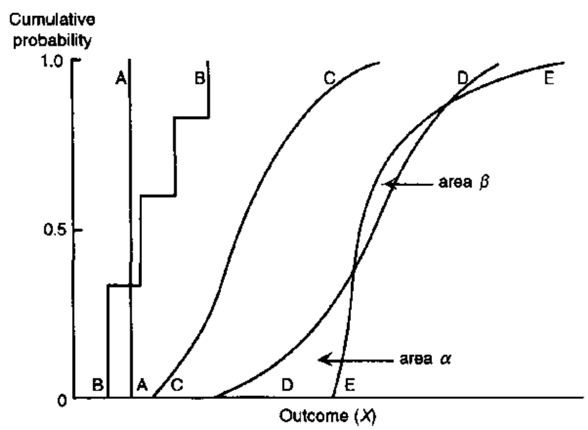

Figure 1: From `http://www.fao.org/`

Definition 1.1. A random variable X stochastically dominates another random variable Y in the first order if $F_X(z) \leq F_Y(z)$ for all z.

Definition 1.2. A random variable X stochastically dominates another random variable Y in the second order if

$$\int_{-\infty}^{\xi} F_X(z)\mathrm{d}z \leq \int_{-\infty}^{\xi} F_Y(z)\mathrm{d}z$$

for all ξ.

2 Risk measures

In many situations where the environment is uncertain, decision making is driven by notions of risk. Risk notions use the language of probability, because in the lands of engineering, uncertainty is modeled by random variables.

Figure 2: From `boston.com`.

Example 2.1 (Course grade)**.** Which of these two choices do you prefer?

1. A guaranteed GPA of 3.0.
2. A random GPA: we flip a coin, with probability 1/2, your GPA is 4.0, and with probability 1/2, your GPA is 2.5.

Notions of risk are used in many discipline, such as economics, politics (policy making), engineering. Each discipline may define distinct notions of risk.

Let X denote the random variable, *e.g.*, corresponding to the end-of-day profit of a newsboy.

- Variance $\mathbb{E}(X - \mathbb{E}X)^2$. This is used in discussions of investment portfolio returns.
- Value-at-risk. Let λ be given and fixed. The value-at-risk with risk-aversion parameter λ is

$$\mathrm{V@R}_\lambda(X) = -q_X(\lambda),$$

  where q_X is the right-continuous quantile function[1] of X.

- Expected shortfall (or *average value-at-risk* or *conditional value-at-risk*). The expected shortfall with risk-aversion parameter λ has the following definition as an integral of V@R:

$$\mathrm{AV@R}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\phi(X)\mathrm{d}\phi.$$

- Entropic risk measure with risk-aversion parameter $\theta > 0$: $\frac{1}{\theta}\log \mathbb{E}(e^{-\theta X})$.

[1]Formally, $q_X(\lambda) = \inf\{x \in \mathbb{R} : F_X(x) > \lambda\}$, where F_X is the distribution function of X.

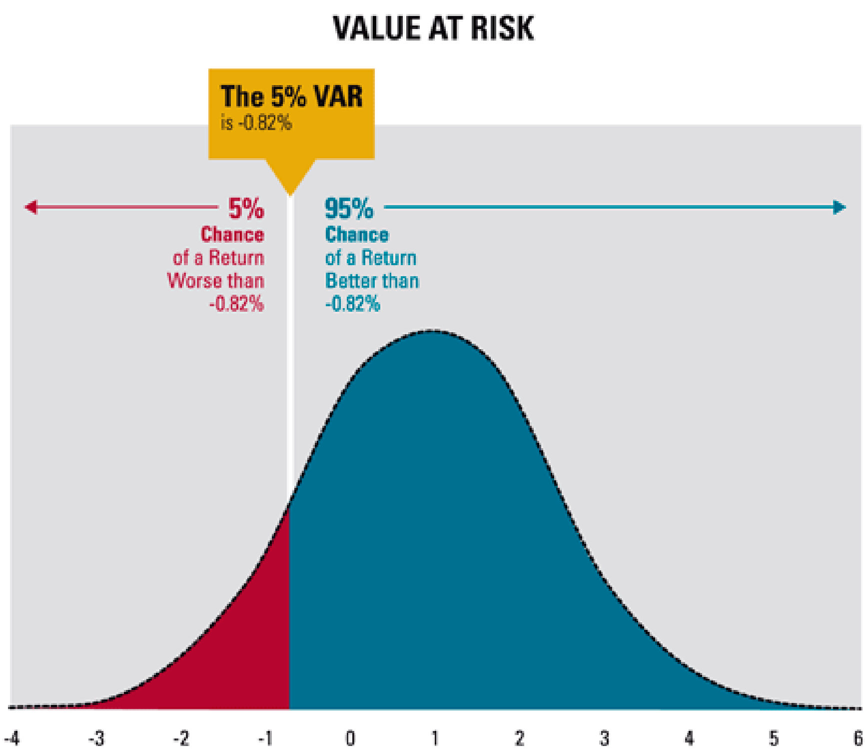

Figure 3: From `advisorone.com`. The x-axis intersept is the VaR, the red area under the probability density function is the risk-aversion parameter λ.

Example 2.2 (Wikipedia explanation for VaR)**.** For example, if a portfolio of stocks has a one-day 5% VaR of CAD 1 million, there is a 0.05 probability that the portfolio will fall in value by more than CAD 1 million over a one day period if there is no trading. Informally, a loss of CAD 1 million or more on this portfolio is expected on 1 day out of 20 days (because of 5% probability).

Remark 1. For normal random variables, both the $\mathrm{V@R}_\lambda$ and the $\mathrm{AV@R}_\lambda$, with probability level $\lambda \geq 0.5$, reduce to a linear combination of mean and standard deviation of the form of $-\mathbb{E}[X] + \beta\sqrt{\mathrm{VAR}[X]}$, where β is only a function of λ (cf. Proof of Proposition 1 of [Rockafellar 2000]).

3 Risk pooling

In this section, we see how a particular decision (to collaborate or not) in a particular inventory management setting can affect profits.

Consider N newsboys selling the same newspaper at different locations. Each newsboy faces a different random demand $D_1, \ldots, D_N$ (assume that these are independent random variables) with known probability distribution $F_1, \ldots, F_N$. Recall from INSE 6290 that, at the start of each day, the optimal stock level of newsboy i is $F_i^{-1}(p/(p+h))$. The total stock of all newsboys is

$$S = \sum_{i=1}^{N} F_i^{-1}(p/(p+h)).$$

Next, consider what happens when the newsboys collaborate by pooling their stocks of newspapers. That is, there is a single cache of newspapers from which newsboys can resupply instantaneously (as long as the cache is not empty); moreover, the newsboys share their revenues equally. Under these assumptions, the optimal cache size at the start of the day is equivalent to a single newsboy facing a demand of $D = D_1 + \ldots + D_N$. Hence, the optimal cache size is

$$\Sigma = F_D^{-1}(p/(p+h)).$$

How does S compare to Σ? This is the first part of your assignment. How does collaboration reflect in the (average) profit of each newsboy?