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2: Preferences	
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We consider the perspective of the decision maker. As seen in the examples from last class, different decision makers have different preferences when faced with uncertainty. Stochastic dominance is easy to apply, but we may not be dealing with well-behaved one-dimensional probability distributions.

1 Preference Relations

Let $\mathcal{X}$ denote the set of possible choices of the decision maker. A preference relation on $\mathcal{X}$ is a binary relation $>$ with the properties:

- Asymmetry: if $x > y$, then $y \not> x$.
- Negative transitivity: if $x > y$ and $z \in \mathcal{X}$, then one of the following hold:
  1. $x > z$,
  2. $z > y$,
  3. $x > z$ and $z > y$.

Given a preference relation $>$ on $\mathcal{X}$, we can construct:

- a weak preference relation $\geq$ defined by $x \geq y \iff y \not> x$,
- an indifference relation $=$ defined by $x = y \iff x \geq y$ and $y \geq x$.

Example 1: Shower first, then brush teeth, versus brush teeth, then shower.
Example 2 ($\geq$): Floss first, then brush teeth, versus brush teeth, then floss.
A preference relation $>$ can sometimes be represented by a function $U : \mathcal{X} \to \mathbb{R}$:

$$x > y \iff U(x) > U(y) \quad \text{for every } x, y \in \mathcal{X}. \tag{1}$$

This function U, if it exits, may not be unique: for every f strictly increasing, we have fU is another representation for $>$.

Example 1.1 (No representation by function U)**.** Some elections with three candidates (Condorcet paradox): Biden $>$ Trump (in general election), Trump $>$ Sanders (in general election), but Sanders $>$ Biden (in primaries among Democrats). Rock, paper, scissors.

1.1 Lotteries

Consider the case where the set $\mathcal{X}$ is a set of probability distributions (lotteries) over a set of outcomes Ω. For example, if there are two outcomes A, B, then the set of lotteries is the set of 2D vectors:

$$\{(p_1, 1-p_1) : p_1 \in [0,1]\}. \tag{2}$$

Example 1.2. In the Rock, paper, scissors example, $\Omega = \{R, P, S\}$. A set of lotteries is of the form

$$\{(1/3, 1/3, 1/3), (1, 0, 0), (0, 1, 0), (0, 1/2, 1/2)\}. \tag{3}$$

Suppose that you play against a friend, what is your preference relation over these lotteries?

More generally, $\mathcal{X}$ can be a set of probability distributions over a larger set Ω, such as the whole real line.

Example 1.3. Lottery ticket, with payoff in $\{0, 10, 100\}$. There are examples of lotteries:

$$(0.9, 0.09, 0.01), \quad (1, 0, 0), \quad (0.2, 0.7, 0.1). \tag{4}$$

1.2 Risk Aversion

We are ready to formally define the notion of risk aversion. Consider the case where $\Omega \subset \mathbb{R}$, where each outcome is a monetary outcome (payoff to the decision maker). If Ω is a different set of outcomes, then we need to first map each outcome to a real number. In turn, the set $\mathcal{X}$ is a set of probability distributions. Let δ_x denote a lottery where outcome x occurs with probability 1 (a vector with value 1 in one element and 0 elsewhere).

Definition 1.1 (Monotone $>$)**.** A preference relation $>$ on $\mathcal{X}$ is monotone if, for $x, y \in \Omega$:

$$x > y \implies \delta_x > \delta_y. \tag{5}$$

Example 1.4 (Monotone)**.** Eating sushi vs brocoli.

Risk aversion captures the intuition that if you prefer receiving the mean of a distribution rather than a random outcome from this distribution.

Definition 1.2 (Risk averse $>$)**.** Let $m(p)$ denote the expected value of the lottery p. A preference relation $>$ is risk averse if

$$\delta_{m(p)} > p \text{ for all } p \in \mathcal{X}, p \neq \delta_{m(p)}. \tag{6}$$

Example 1.5. Let $\Omega = \{100, 800, 900, 4000\}$. Buy plane ticket now for the expected price $m(p)$, or buy it just before flying for a random cost distributed according to p? Buy the ticket today: $p = (0, 0, 1, 0)$. Buy the ticket tomorrow: $p = (0, 0.5, 0.5, 0)$. But the ticket last minute: $p = (0.5, 0, 0, 0.5)$. Using everday language, we would say that a decision maker is risk averse if he or she chooses to buy ticket today or tomorrow. But to check if the decision maker is risk averse in the formal sense, we have to check the definition.

Note that checking the risk aversion property is cumbersome: you have to check one binary relation for each pair of lotteries in $\mathcal{X}$. Is there a shortcut, for instance preferences with a representation function?

2 Von Neumann-Morgenstern Representation

Suppose that $\Omega = \{\omega_1, \dots, \omega_d\}$ is finite and that $\mathcal{X}$ is convex. A lottery $p \in \mathcal{X}$ is a probability distribution on Ω:

$$p = (p_1, \dots, p_d), p_i \geq 0, \sum_i p_i = 1. \tag{7}$$

A representation U for a preference relation $>$ on $\mathcal{X}$ is a Von Neumann-Morgenstern Representation if there exists a function $u : \Omega \to \mathbb{R}$ such that

$$U(p) = \sum_{i=1}^{d} p_i u(\omega_i) \quad \text{for all } p \in \mathcal{X}. \tag{8}$$

We also write

$$U(p) \triangleq \mathbb{E}_{Z \sim p} u(Z). \tag{9}$$

Some preference relations have a representation U, others don't as we have seen above. Further, some representations have a Von Neumann-Morgenstern Representation (e.g., risk-neutral decision maker with the identity function as u). What are examples of representations without a Von Neumann-Morgenstern Representation u?

Consider the case where $\Omega = \{0, 2400, 2500\}$, and the following lotteries:

$$p = (0, 1, 0) \quad q = (0.01, 0.66, 0.33) \tag{10}$$
$$p' = (0.66, 0.34, 0) \quad q' = (0.67, 0, 0.33). \tag{11}$$

Consider a preference $>$ such that

$$p > q \text{ and } q' > p'. \tag{12}$$

Suppose that this preference $>$ has a representation U, then

$$U(p) > U(q) \text{ and } U(q') > U(p'). \tag{13}$$

Suppose, on the contrary, that U has a Von Neumann-Morgenstern Representation u, then

$$U(\frac{1}{2}p + \frac{1}{2}q') = \sum_{i=1}^{d}(\frac{1}{2}p + \frac{1}{2}q')u(\omega_i) \tag{14}$$

$$(\text{by linearity}) = \frac{1}{2}\sum_{i=1}^{d} p_i u(\omega_i) + \frac{1}{2}\sum_{i=1}^{d} q'_i u(\omega_i) \tag{15}$$

$$(\text{definition}) = \frac{1}{2}U(p) + \frac{1}{2}U(q') \tag{16}$$

$$(\text{by (13)}) > \frac{1}{2}U(q) + \frac{1}{2}U(p') \tag{17}$$

$$(\text{definition}) = U(\frac{1}{2}q + \frac{1}{2}p'). \tag{18}$$

However, we also have by definition:

$$\frac{1}{2}p + \frac{1}{2}q' = (\frac{0.67}{2}, \frac{1}{2}, \frac{0.33}{2}) = \frac{1}{2}q + \frac{1}{2}p', \tag{19}$$

and hence $U(\frac{1}{2}p + \frac{1}{2}q') = U(\frac{1}{2}q + \frac{1}{2}p')$. This leads to a contradiction.

3 References

- Chapter 2 of textbook.